

# DESCENT IN ALGEBRAIC $K$ -THEORY AND A CONJECTURE OF AUSONI-ROGNES

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**ABSTRACT.** Let  $A \rightarrow B$  be a  $G$ -Galois extension of rings, or more generally of  $\mathbb{E}_\infty$ -ring spectra in the sense of Rognes. A basic question in algebraic  $K$ -theory asks how close the map  $K(A) \rightarrow K(B)^{hG}$  is to being an equivalence, i.e., how close algebraic  $K$ -theory is to satisfying Galois descent. An elementary argument with the transfer shows that this equivalence is true rationally in most cases of interest. Motivated by the classical descent theorem of Thomason, one also expects such a result after periodic localization.

We formulate and prove a general result which enables one to promote rational descent statements as above into descent statements after periodic localization. This reduces the localized descent problem to establishing an elementary condition on  $K_0(-) \otimes \mathbb{Q}$ . As applications, we prove various descent results in the periodic localized  $K$ -theory,  $TC$ ,  $THH$ , etc. of structured ring spectra, and verify several cases of a conjecture of Ausoni and Rognes.

## 1. INTRODUCTION

**1.1. Motivation.** Let  $X$  be a noetherian scheme. A subtle and important invariant of  $X$  is given by the *algebraic  $K$ -theory* groups  $\{K_n(X)\}_{n \geq 0}$ . As  $X$  varies, the groups  $\{K_n(X)\}$  behave something like a cohomology theory in  $X$ . For example, they form a contravariant functor in  $X$  and there is an analog of the classical Mayer-Vietoris sequence thanks to the localization properties of algebraic  $K$ -theory [TT90, Thm. 10.3]. A highbrow formulation of this property is that the groups  $\{K_n(X)\}$  arise as the homotopy groups of a *spectrum*  $K(X)$ , and that the contravariant functor

$$K(-): \text{Sch}^{op} \rightarrow \text{Sp}_{\geq 0}$$

forms a sheaf of *connective spectra* on the Zariski site of  $X$ .<sup>1</sup>

As is well-known, however, the Zariski topology of  $X$  is too coarse to have a strong analogy with algebraic topology: a more appropriate topology is given by the *étale* topology. One might hope that  $K$  is a sheaf (i.e., behaves ‘like a cohomology theory’) for the étale topology; if so, one could then hope for a local-to-global spectral sequence (an analog of the Atiyah-Hirzebruch spectral sequence for topological  $K$ -theory) beginning with étale cohomology and ultimately converging to algebraic  $K$ -theory. Indeed, the convergence properties of such a spectral sequence are the subject of the Quillen-Lichtenbaum conjecture [Lic73, Qui75], which is a consequence of the Rost-Voevodsky Norm Residue theorem (see [Kol15] for a recent survey).

The problem is that  $K$ -theory is not a sheaf for the étale topology. If  $E \rightarrow F$  is a  $G$ -Galois extension of fields, one has a  $G$ -action on  $K(F)$  and a natural map

$$(1.1) \quad K(E) \rightarrow K(F)^{hG},$$

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<sup>1</sup>To obtain a sheaf of spectra, one has to work with the non-connective version  $\mathbb{K}$  of  $K$ -theory.

but this need not be an equivalence, contradicting étale descent. In fact, since algebraic  $K$ -theory satisfies Nisnevich descent (cf. [Nis89] and [TT90, Thm. 10.8]), the failure of descent along Galois extensions of commutative rings is the *only* obstruction to satisfying étale descent [Lur11, Cor. 4.24].

In the foundational paper [Tho85] and in the later extension [TT90], Thomason showed that these problems disappear after a localization, after which maps of the form (1.1) become equivalences. Specifically, let  $X$  be a scheme where a fixed prime number  $\ell$  is invertible and suppose  $X$  contains the  $\ell$ th roots of unity. If  $\ell \geq 5$  (as we assume for simplicity), there is a natural element  $\beta \in \pi_2(K(X)/\ell)$  called the *Bott element*. Under these assumptions,  $K(X)/\ell$  is naturally a ring spectrum up to homotopy and one can form the localization  $(K(X)/\ell)[\beta^{-1}]$ .

**Theorem 1.2** (Thomason [Tho85, Thm. 2.45] and [TT90, Thm. 11.5]). Suppose  $X$  is a noetherian scheme of finite Krull dimension over  $\mathbb{Z}[1/\ell, \mu_\ell]$ , satisfying Thomason’s technical hypothesis on the existence of ‘Tate-Tsen filtrations’ on the residue fields of uniformly bounded length.<sup>2</sup> Then the functor  $Y \mapsto (K(Y)/\ell)[\beta^{-1}]$  is a sheaf of spectra on the small étale site of  $X$ . Moreover, there exists a descent spectral sequence

$$H_{\text{ét}}^s(X, \tilde{\pi}_{2t}K/\ell[\beta^{-1}]) \cong H_{\text{ét}}^s(X, \mathbb{Z}/\ell(t)) \implies \pi_{2t-s}(K(X)/\ell)[\beta^{-1}].$$

Thomason observed that there is another construction of Bott inverted  $K$ -theory [Tho85, §A.14] that makes no reference to Bott elements. One can first form mod  $\ell$   $K$ -theory by smashing with the Moore spectrum  $S/\ell$  and then obtain Bott-periodic  $K$ -theory by inverting the Adams self-map  $v$  of the Moore spectrum. In the setting above we have an equivalence:

$$K(-)/\ell[\beta^{-1}] \simeq K(-) \wedge S/\ell[v^{-1}].$$

Now if we invert those maps which become an equivalence after smashing with  $T(1) := S/\ell[v^{-1}]$ , we obtain a localization functor  $L_{T(1)}$ . It follows from Thomason’s theorem that  $L_{T(1)}K(-)$  satisfies étale descent under the above hypotheses. We will generalize this statement below to the world of structured ring spectra.

**1.2. Extending Thomason’s rational descent argument.** As the above notation indicates, there is an infinite family of such telescopic localization functors  $\{L_{T(n)}\}$  indexed over the integers  $n \geq 0$  and primes  $\ell$ . When  $n = 0$ ,  $T(0)$ -equivalences are precisely rational equivalences (for every prime  $\ell$ ). Thomason observed that proving the rational analog of Theorem 1.2 is actually quite easy and reduces to a transfer argument for finite Galois extensions [Tho85, Thm. 2.15].

We will show that Thomason’s argument for the rational case actually implies étale descent for  $L_{T(1)}K(-)$  by exploiting the fact that the algebraic  $K$ -theory of commutative rings does not just take values in homotopy commutative ring spectra, but rather in (vastly more structured)  $\mathbb{E}_\infty$ -ring spectra. In this setting we can apply the May nilpotence conjecture [MNN15b].

Since Thomason’s argument is so simple and central to the motivation of this paper, we will recall it in a modernized form. Consider the case of a  $G$ -Galois extension of fields  $A \rightarrow B$ . Using techniques of Merling [Mer15], Barwick [Bar15], and Barwick-Glasman-Shah [BGS15], one constructs an  $\mathbb{E}_\infty$ -algebra  $R = K(B; G)$  in the  $\infty$ -category of  $G$ -spectra such that for any subgroup  $H \leq G$ , we have  $R^H = K(B; G)^H = K(B^{hH})$ . The descent comparison map now becomes the classical comparison map

$$(1.3) \quad K(A) = R^G \rightarrow R^{hG} = K(B)^{hG}$$

<sup>2</sup>This implies that the residue fields of  $X$  have uniformly bounded  $\ell$ -torsion Galois cohomological dimension. To the authors’ knowledge, no counterexample to the converse implication is known.

from fixed points to homotopy fixed points. It is a general fact about  $G$ -spectra that the fiber of this comparison map is a module over the *cofiber*  $C = K(A)/K(B)_{hG}$  of the transfer map

$$K(B)_{hG} \rightarrow K(A),$$

which arises from restriction of scalars along  $A \rightarrow B$ . The ring spectrum  $C$  has an  $\mathbb{E}_\infty$ -structure: in the language of equivariant stable homotopy theory, we have  $C = (R \wedge \widetilde{EG})^G$ . Now for Galois extensions of fields, the induced map

$$(1.4) \quad K_0(B) \rightarrow K_0(A)$$

is the map  $\mathbb{Z} \xrightarrow{|G|} \mathbb{Z}$ . It follows that  $\pi_0 C \simeq \mathbb{Z}/|G|$ , so that  $C$  is in particular rationally trivial. We use the  $C$ -module structure on the fiber of (1.3) to obtain Thomason's equivalences:

$$(1.5) \quad K(A) \otimes \mathbb{Q} \xrightarrow{\sim} (K(B)^{hG}) \otimes \mathbb{Q} \xrightarrow{\sim} (K(B) \otimes \mathbb{Q})^{hG}.$$

These equivalences are direct consequence of the ring structures and the rational surjectivity of (1.4). Moreover, the equivalences in (1.5) *imply* the rationalized transfer map  $K_{hG}(B) \otimes \mathbb{Q} \rightarrow K(A) \otimes \mathbb{Q}$  is an equivalence which, in turn, implies the surjectivity in the rationalization of (1.4). So the rational surjectivity of (1.4) is a *necessary and sufficient* condition for the descent equivalences in (1.5).

We now build on Thomason's argument by using the  $\mathbb{E}_\infty$ -structure on  $C$ . The proof of the May conjecture from [MNN15b] says the rational triviality of  $C$  is equivalent to the triviality of  $L_T C$  for *every* telescopic localization of  $C$ . It follows that the rational surjectivity of the transfer is equivalent to the  $L_{T(0)} = \mathbb{Q}$ -equivalences in (1.5) which are, in turn, equivalent to those equivalences after we replace  $L_{T(0)}$  with any telescopic localization.

**1.3. Methods.** In order to consider non-Galois extensions, we will not use the technology of equivariant algebraic  $K$ -theory or the language of  $G$ -spectra in this paper. Instead we will work with the symmetric monoidal stable  $\infty$ -category  $\text{Mot}_A$  of non-commutative  $A$ -linear motives developed in [BGT13, BGT14, HSS15] as a replacement for the equivariant stable homotopy category. We will now include a brief sketch of our methods.

By construction, non-commutative motives form the universal stable  $\infty$ -category for studying weakly additive invariants (see Definition 3.11) such as  $K$ -theory,  $THH$ , or  $TC$ . To be more specific, let  $E$  be such a functor valued in spectra. Then the restriction of  $E$  to a functor on commutative  $A$ -algebras canonically factorizes as:  $\text{CAlg}(A) \xrightarrow{[-]} \text{Mot}_A \xrightarrow{\tilde{E}} \text{Sp}$ , where  $[-]$  takes tensor products of  $A$ -algebras to tensor products in  $\text{Mot}_A$  and  $\tilde{E}$  is an exact functor uniquely determined by  $E$ . This allows us to restrict our attention to exact functors out of motives.

Consider the full subcategory  $\mathcal{I} \subset \text{Mot}_A$  of all those  $M \in \text{Mot}_A$  for which the augmented cosimplicial object

$$M \rightarrow M \otimes [B] \rightrightarrows M \otimes [B] \otimes [B] \Rrightarrow \cdots$$

becomes a limit diagram after applying *any* exact functor. Since  $\mathcal{I}$  is a  $\otimes$ -ideal, we know that the monoidal unit  $[A]$  belongs to  $\mathcal{I}$  precisely when the symmetric monoidal Verdier quotient  $\text{Mot}_A/\mathcal{I}$  is trivial<sup>3</sup>, which is, in turn, equivalent to the triviality of the  $\mathbb{E}_\infty$ -ring spectrum  $R = \text{Hom}_{\text{Mot}_A/\mathcal{I}}([A], [A])$ . This ring spectrum will play the same role as  $C$  did in Thomason's argument above.

Using the universal properties of  $K$ -theory we obtain a ring map  $K_0(A) \rightarrow \pi_0 R$  which, since  $[B]$  belongs to  $\mathcal{I}$ , sends anything in the image of a 'transfer map'  $K_0(B) \rightarrow K_0(A)$  (which makes

<sup>3</sup>To be precise, one should first pass to suitably small subcategories before taking this quotient.

sense if there is a morphism  $[B] \rightarrow [A]$  to zero. In particular, if there is a surjective transfer map  $K_0(B) \twoheadrightarrow K_0(A)$ , then  $[A] \in \mathcal{I}$ .

For our localized descent results we want to show that  $[A]$  lies in  $\mathcal{I}$  ‘up to telescopic localization’, which will be equivalent to asking for the desired equivalences

$$L_T \tilde{E}([A]) \xrightarrow{\sim} L_T \text{Tot} \left( \tilde{E}([B]^{\otimes \bullet+1}) \right) \xrightarrow{\sim} \text{Tot} \left( L_T \tilde{E}([B]^{\otimes \bullet+1}) \right)$$

for any exact functor  $\tilde{E}$  and any telescopic localization  $L_T$ . This will be formalized in Section 2 by saying that  $[A]$  lies in the  $\varepsilon$ -enlargement  $\mathcal{I}_\varepsilon$  of  $\mathcal{I}$ .

To see that  $[A]$  lies in  $\mathcal{I}_\varepsilon$ , we no longer need to check the triviality of  $R$ . Instead, we need to check the triviality of each of the telescopic localizations of  $R$ . Since  $R$  is an  $\mathbb{E}_\infty$ -ring spectrum, the solution to May’s conjecture implies that this condition is equivalent to the rational triviality of  $\pi_0 R$ . We can now argue as above and see that this condition follows from the existence of a rationally surjective transfer map  $K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$ .

#### 1.4. The $K$ -theory of structured ring spectra and a conjecture of Ausoni and Rognes.

The above argument shows that one can drop some of Thomason’s technical hypotheses and still obtain étale descent for  $L_{T(n)} K(-)$ , for every  $n \geq 0$  and implicit prime  $\ell$ . On the one hand, when  $n \geq 2$ , a result of Mitchell [Mit90] shows  $L_{T(n)} K(X)$  is trivial for every scheme  $X$ , so our argument provides no information. On the other hand, the arguments above are very robust and one can hope that they will generalize to other contexts.

Waldhausen [Wal84] proposed such a context, namely that the  $K$ -theory of rings could be extended to ‘brave new rings’ (now called structured ring spectra). This proposal has been realized in the work of many people (cf., e.g., [EKMM97, BGT13]) and numerous tools have been developed for this generalization. This has led to deep calculations in certain important cases. We refer to [Rog14] for a recent survey.

To further extend the analogy with algebra, Rognes [Rog08] formulated a notion of a Galois extension of  $\mathbb{E}_\infty$ -ring spectra generalizing the classical notion in commutative algebra. A fundamental example of such an extension is the complexification map  $KO \rightarrow KU$  from real to complex topological  $K$ -theory. There are higher chromatic analogs of this example coming from Lubin-Tate spectra and the theory of topological modular forms.

In this setting, Ausoni and Rognes made the following descent conjecture, which we can view as the higher chromatic analog of the statement of Theorem 1.2:

**Conjecture 1.6** (Ausoni and Rognes [AR08]). Let  $A \rightarrow B$  be a  $K(n)$ -local  $G$ -Galois extension of  $\mathbb{E}_\infty$ -rings. Let  $T(n+1)$  be a telescope of a  $v_{n+1}$ -self map of a type  $(n+1)$ -complex. Then the map

$$T(n+1) \wedge K(A) \rightarrow T(n+1) \wedge K(B)^{hG}$$

is an equivalence.

The following main result, which is proven in the body of the paper as Theorem 5.1, will imply several important cases of Conjecture 1.6, but also applies to non-Galois extensions.

**Theorem 1.7.** Suppose  $A \rightarrow B$  is a morphism of  $\mathbb{E}_\infty$ -rings such that  $B$  is a perfect  $A$ -module and such that the induced rationalized restriction of scalars map  $K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$  is surjective. Let  $E(-)$  be either algebraic  $K$ -theory  $K(-)$ , non-connective algebraic  $K$ -theory  $\mathbb{K}(-)$ ,  $THH(-)$ , or  $TC(-)$  and let  $L_T$  denote one of the following ‘periodic localization’ functors:  $L_{T(n)}$ ,  $L_{K(n)}$ ,  $L_n^f$ , or  $L_n$  (taken at an implicit prime  $\ell$  for some  $n \geq 0$ ).

Then the map

$$(1.8) \quad E(A) \rightarrow \mathrm{Tot} \left( E(B) \rightrightarrows E(B \otimes_A B) \rightrightarrows \dots \right)$$

becomes an equivalence after  $L_T$ -localization (which can be performed either inside or outside the totalization). Moreover, the associated Tot/Čech-spectral sequence collapses at a finite page with a horizontal vanishing line.

In particular, if  $A \rightarrow B$  is a  $G$ -Galois extension satisfying the above hypothesis on  $K_0(-) \otimes \mathbb{Q}$ , then the Ausoni-Rognes Conjecture 1.6 holds for this extension. In this case, the associated spectral sequence is the homotopy fixed point spectral sequence:

$$H^s(G; \pi_t L_T E(B)) \Rightarrow \pi_{t-s} L_T E(A).$$

Theorem 1.7 reduces the localized descent problem to a question about  $K_0(-) \otimes \mathbb{Q}$ . This condition is relatively accessible and can be checked in many examples of interest:

**Theorem 1.9.** The hypotheses of Theorem 1.7 are satisfied for the following maps of  $\mathbb{E}_\infty$ -ring spectra:

- Any finite étale cover  $A \rightarrow B$  (see Proposition 5.4 for a more general condition).
- The complexification maps of topological  $K$ -theory spectra:  $KO \rightarrow KU$  or  $ko \rightarrow ku$  (Examples 5.9 and 5.30).
- The  $G$ -Galois extensions  $E_n^{hG} \rightarrow E_n$  where  $G \subset \mathbb{G}_n$  is a finite subgroup of the extended Morava stabilizer group (Corollary B.4).
- Any finite  $G$ -Galois extension of the following variations on topological modular forms:  $TMF[1/n]$ ,  $Tmf_0(n)$ , or  $Tmf_1(n)$  (Theorems 5.24 and 5.27), where  $n \geq 1$ .
- The extension  $tmf[1/3] \rightarrow tmf_1(3)$  of connective topological modular forms (Example 5.31).

For extensions defined by certain higher real  $K$ -theories one has sharper results (see Theorem 5.10).

**1.5. Further remarks.** In the algebraic  $K$ -theory of *connective* ring spectra, the theory of *trace methods* is a fundamental tool. Using it, one can try to answer the above question by combining Thomason's results together with comparisons with topological cyclic homology. However, the most interesting Galois extensions arise from nonconnective ring spectra (in fact, all Galois extensions of connective ring spectra can be determined in terms of pure algebra by [Mat16, Thm. 6.17] and [MM15, Ex. 5.5]). As a result, our approach in this paper is completely different.

We also emphasize that our methods *do not* recover Thomason's spectral sequence, whose  $E_2$ -term is the sheaf cohomology. The convergence of that spectral sequence requires étale *hyperdescent*, rather than the descent result we prove. It seems to us that any hyperdescent statement for non-rational localizations will require additional tools.

However, if we ignore the issue of sheaf vs. hypersheaf, then our methods show that none of the technical hypotheses on  $X$  imposed in Thomason's theorem are actually necessary. Thus, we do obtain new descent results even in the algebraic  $K$ -theory of discrete rings; see also Example 5.5.

Finally, we remark that another crucial aspect of Thomason's spectral sequence is the identification of the étale sheafified homotopy groups of  $K/\ell(-)[\beta^{-1}]$  with the  $\mathbb{Z}/\ell(n)$  spaced out in even degrees, so that the  $E_2$ -term can be written explicitly. This identification follows from the Gabber-Suslin rigidity theorem, which we do not know the analog for in the setting of structured ring spectra and higher chromatic localizations.

**1.6. Outline.** In Section 2 we introduce the notion of  $\varepsilon$ -objects and  $\varepsilon$ -equivalences. This robust theory allows us to formulate rigorously what it means for a homotopy limit to quickly converge modulo objects which are invisible to chromatic homotopy theory. In Section 3 we review the relative theory of non-commutative motives as developed by [BGT13, HSS15]. This section is technical and can be skipped on a first reading. The only minor variation in our treatment allows us to consider additive invariants which do not commute with all filtered colimits (such as  $TC$ ).

In Section 4, we show how the nilpotence criterion for  $\mathbb{E}_\infty$ -rings of [MNN15b] provides a simple criterion for establishing descent results. In Section 5 we establish our aforementioned examples and prove that periodically localized  $K$ -theory is a sheaf for the finite flat topology on  $\mathbb{E}_\infty$ -ring spectra. As a corollary, we obtain in Appendix A an analog for spectral algebraic spaces of Thomason's result that the  $K$ -theory of schemes satisfies étale descent after periodic localization. Appendix B by Meier, Naumann, and Noel shows the existence of finite even complexes with specific Morava  $K$ -theories. This might be of independent interest, but serves the immediate purpose of establishing descent for the algebraic  $K$ -theory of higher real  $K$ -theories.

### Notation.

- (1) We will write  $\text{Cat}_\infty^{\text{perf}}$  for the  $\infty$ -category of idempotent-complete, small stable  $\infty$ -categories and exact functors between them. We recall that  $\text{Cat}_\infty^{\text{perf}}$  is a presentable, symmetric monoidal  $\infty$ -category under the Lurie tensor product. We refer to [BGT13] for an account of the general features of  $\text{Cat}_\infty^{\text{perf}}$ .
- (2) We let  $\text{CAlg}(\text{Cat}_\infty^{\text{perf}})$  denote the  $\infty$ -category of commutative algebra objects in  $\text{Cat}_\infty^{\text{perf}}$ : equivalently, this is the  $\infty$ -category of small symmetric monoidal, idempotent-complete stable  $\infty$ -categories  $(\mathcal{C}, \otimes, \mathbf{1})$  where  $\otimes$  is exact in each variable. The morphisms in  $\text{CAlg}(\text{Cat}_\infty^{\text{perf}})$  are the symmetric monoidal exact functors. Given an object  $X \in \mathcal{C}$ , we will write  $\pi_k(X) = \pi_k \text{Hom}_{\mathcal{C}}(\mathbf{1}, X)$ .
- (3) We will let  $\mathcal{P}r_{\text{st}}^L$  denote the category of presentable, stable  $\infty$ -categories and cocontinuous functors between them, with the Lurie tensor product.
- (4) We will also need to consider not necessarily small, idempotent-complete stable  $\infty$ -categories and exact functors between them, and we denote the  $\infty$ -category of such by  $\widehat{\text{Cat}}_\infty^{\text{perf}}$ . Thus  $\widehat{\text{Cat}}_\infty^{\text{perf}}$  includes both small and presentable idempotent-complete, stable  $\infty$ -categories.
- (5) Although due to set-theoretic technicalities we will not consider  $\widehat{\text{Cat}}_\infty^{\text{perf}}$  as a symmetric monoidal  $\infty$ -category, we will abuse notation and write  $\text{CAlg}(\widehat{\text{Cat}}_\infty^{\text{perf}})$  for the  $\infty$ -category of symmetric monoidal, stable, idempotent-complete  $\infty$ -categories with a biexact tensor product, and symmetric monoidal exact functors between them.

We write  $\text{Sp}$  for the  $\infty$ -category of spectra and  $\text{Sp}^\omega \subset \text{Sp}$  for the subcategory of finite spectra. Given an  $\mathbb{E}_\infty$ -ring spectrum  $R$ , we write  $\text{Perf}(R)$  for the  $\infty$ -category of perfect (i.e., compact)  $R$ -modules. We will write  $\mathbb{D}$  for the dual of an object (e.g., the Spanier-Whitehead dual of a finite spectrum). Given objects  $X, Y$  of a stable  $\infty$ -category  $\mathcal{C}$ , we will write  $\text{Hom}_{\mathcal{C}}(X, Y) \in \text{Sp}$  for the mapping *spectrum*.

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## 2. $\varepsilon$ -NILPOTENCE

Let  $\mathcal{C} \in \widehat{\text{Cat}}_{\infty}^{\text{perf}}$  be a (not necessarily small) stable, idempotent-complete  $\infty$ -category and let  $\mathcal{T} \subset \mathcal{C}$  be a full subcategory. Recall that  $\mathcal{T}$  is called *thick* if  $\mathcal{T}$  is a stable subcategory which is also idempotent-complete, so  $\mathcal{T} \in \widehat{\text{Cat}}_{\infty}^{\text{perf}}$ . Using the nilpotence theorem [DHS88], Hopkins and Smith gave in [HS98, Thm. 7] a complete classification of thick subcategories of the category of finite spectra. The classification of thick subcategories in general is a problem that has been further studied in many different contexts and is closely related to questions of nilpotence.

A key technique of this paper uses a specific enlargement  $\mathcal{T}_{\varepsilon}$  of a thick subcategory  $\mathcal{T} \subset \mathcal{C}$ . Roughly speaking,  $\mathcal{T}_{\varepsilon}$  is the maximal enlargement of  $\mathcal{T}$  which is no different from  $\mathcal{T}$  from the point of view of any periodic localization. We will discuss some important examples of this construction and, ultimately, a basic criterion that enables one to check whether an object belongs to  $\mathcal{T}_{\varepsilon}$  by performing a *rational* calculation (Theorem 4.2). All the results in the present section are fairly formal consequences of the nilpotence technology of [DHS88, HS98].

**2.1.  $\varepsilon$ -enlargements.** In this subsection, we make the basic definition. In the remaining subsections, we will explore this further for specific choices of thick subcategories.

**Definition 2.1.** Let  $\mathcal{C} \in \widehat{\text{Cat}}_{\infty}^{\text{perf}}$  and let  $\mathcal{T} \subset \mathcal{C}$  be a thick subcategory. We will define several enlargements of  $\mathcal{T}$ . Recall that  $\mathcal{C}$  is naturally tensored over finite spectra.

- (1) Given a finite spectrum  $F$ , we define  $\mathcal{T}_F$  to be the smallest thick subcategory of  $\mathcal{C}$  containing  $\mathcal{T}$  and  $\{F \wedge C\}_{C \in \mathcal{C}}$ .
- (2) Let  $\Sigma$  be a finite set of prime numbers. We define thick subcategories  $\mathcal{T}_{\varepsilon, \Sigma}$  via

$$\mathcal{T}_{\varepsilon, \Sigma} = \bigcap_F \mathcal{T}_F,$$

where  $F$  ranges over all finite spectra whose  $p$ -localization is nontrivial for every  $p \in \Sigma$ .

- (3) Finally, we define the thick subcategory  $\mathcal{T}_{\varepsilon}$  via

$$\mathcal{T}_{\varepsilon} = \bigcup_{\Sigma} \mathcal{T}_{\varepsilon, \Sigma},$$

as  $\Sigma$  ranges over all finite sets of prime numbers. We will call this the  $\varepsilon$ -enlargement of  $\mathcal{T}$ .

The process of  $\varepsilon$ -enlargement interacts well with exact functors.

**Proposition 2.2.** Suppose  $G: \mathcal{C} \rightarrow \mathcal{D}$  is a morphism in  $\widehat{\text{Cat}}_{\infty}^{\text{perf}}$  (i.e., an exact functor). Suppose  $\mathcal{T} \subset \mathcal{C}$  and  $\mathcal{T}' \subset \mathcal{D}$  are thick subcategories and  $G(\mathcal{T}) \subset \mathcal{T}'$ . Then  $G(\mathcal{T}_{\varepsilon}) \subset \mathcal{T}'_{\varepsilon}$ . Furthermore, for each finite set of prime numbers  $\Sigma$ , we have  $G(\mathcal{T}_{\varepsilon, \Sigma}) \subset \mathcal{T}'_{\varepsilon, \Sigma}$ .

*Proof.* Let  $F$  be any finite spectrum. One shows easily that  $G(\mathcal{T}_F) \subset \mathcal{T}'_F$  (because  $G$  commutes with smashing with the finite spectrum  $F$ ) and then the remaining assertions follow formally by taking intersections and unions.  $\square$

**Definition 2.3.** Let  $\mathcal{C} \in \widehat{\text{Cat}}_\infty^{\text{perf}}$ . Suppose  $\mathcal{T} = \{0\}$ . Then we write  $\text{Nil}_\varepsilon(\mathcal{C}) \stackrel{\text{def}}{=} \mathcal{T}_\varepsilon$  and call it the subcategory of  $\varepsilon$ -objects of  $\mathcal{C}$ . In the case  $\mathcal{C} = \text{Sp}$ , we will call this the subcategory of  $\varepsilon$ -spectra. We will say that a morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  is an  $\varepsilon$ -equivalence if the cofiber is an  $\varepsilon$ -object. Finally, we also write  $\text{Nil}_{\varepsilon, \Sigma}(\mathcal{C})$  for  $\mathcal{T}_{\varepsilon, \Sigma}$  for any finite set  $\Sigma$  of prime numbers.

Our next result will be Proposition 2.5, which explains that the general formation of  $\mathcal{T}_\varepsilon$  can be reduced to the special case that  $\mathcal{T} = \{0\}$ , using Verdier quotients.

Suppose now that  $\mathcal{C}$  is *small*, i.e.,  $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$ . Suppose  $\mathcal{T} \subset \mathcal{C}$  is a thick subcategory. Then we recall that we can form the (idempotent-complete) Verdier quotient  $\mathcal{C}/\mathcal{T} \in \text{Cat}_\infty^{\text{perf}}$  receiving an exact functor from  $\mathcal{C}$  (cf. [BGT13, §5.1] for a treatment; in particular,  $\mathcal{C}/\mathcal{T}$  is the pushout  $\mathcal{C} \sqcup_{\mathcal{T}} 0$  in  $\text{Cat}_\infty^{\text{perf}}$ ). An object of  $\mathcal{C}$  belongs to  $\mathcal{T}$  if and only if its image in the Verdier quotient vanishes.

Note that Proposition 2.2 implies that  $\varepsilon$ -objects are preserved by any exact functor in  $\text{Cat}_\infty^{\text{perf}}$ . Applying this to the functor  $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{T}$  shows that  $\mathcal{T}_\varepsilon$  is mapped to  $\text{Nil}_\varepsilon(\mathcal{C}/\mathcal{T})$ , and to see the converse, we first need an elementary lemma whose proof we leave to the reader.

**Lemma 2.4.** Let  $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$  and let  $\mathcal{T} \subset \mathcal{C}$  be a thick subcategory. Then  $\mathcal{T} = \mathcal{C} \cap \text{Ind}(\mathcal{T}) \subset \text{Ind}(\mathcal{C})$ .

**Proposition 2.5.** Let  $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$  and let  $\mathcal{T} \subset \mathcal{C}$  be a thick subcategory. Then an object  $X \in \mathcal{C}$  belongs to  $\mathcal{T}_\varepsilon$  if and only if its image in  $\mathcal{C}/\mathcal{T}$  is an  $\varepsilon$ -object.

*Proof.* The ‘only if’ implication follows from Proposition 2.2, applied to the canonical functor  $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{T}$ . For the converse, suppose  $\Sigma$  is a finite set of prime numbers and the image  $\overline{X} \in \mathcal{C}/\mathcal{T}$  of  $X$  belongs to  $\text{Nil}_{\varepsilon, \Sigma}(\mathcal{C}/\mathcal{T})$ . We show that  $X \in \mathcal{T}_{\varepsilon, \Sigma}$ .

Let  $F$  be any finite spectrum such that  $F_{(p)} \neq 0$  for each  $p \in \Sigma$ . We need to show that  $X$  belongs to  $\mathcal{T}_F$ . Equivalently, by Lemma 2.4, we need to show that  $X$  belongs to  $\text{Ind}(\mathcal{T}_F) \subset \text{Ind}(\mathcal{C})$ . In  $\text{Ind}(\mathcal{C})$  we can localize to obtain a cofiber sequence,

$$X' \rightarrow X \rightarrow X'',$$

where  $X' \in \text{Ind}(\mathcal{T}) \subset \text{Ind}(\mathcal{C})$  while  $\text{Hom}_{\text{Ind}(\mathcal{C})}(T, X'') = 0$  for  $T \in \mathcal{T}$ . Thus, it suffices to show that  $X''$  belongs to the localizing subcategory generated by  $\{F \wedge Y\}_{Y \in \mathcal{C}}$ .

Finally, we recall (cf. [BGT13, §5.1]) that the collection of  $Z \in \text{Ind}(\mathcal{C})$  with  $\text{Hom}_{\text{Ind}(\mathcal{C})}(T, Z) = 0$  for each  $T \in \mathcal{T}$  can be identified with  $\text{Ind}(\mathcal{C}/\mathcal{T})$  and  $X''$  can be identified with the image  $\overline{X} \in \mathcal{C}/\mathcal{T} \subset \text{Ind}(\mathcal{C}/\mathcal{T})$ . So, our assumption that  $\overline{X} \in \text{Nil}_{\varepsilon, \Sigma}(\mathcal{C}/\mathcal{T})$  now shows that  $X''$  belongs to the localizing subcategory as desired.  $\square$

We next observe that being an  $\varepsilon$ -object can be checked locally at one prime at a time. The proof of the next proposition is left to the reader.

**Proposition 2.6.** Let  $\mathcal{C} \in \widehat{\text{Cat}}_\infty^{\text{perf}}$ . Fix an object  $X \in \mathcal{C}$ . Then  $X \in \text{Nil}_\varepsilon(\mathcal{C})$  if and only if the following conditions are satisfied:

- (1) There exists  $N \in \mathbb{Z}_{>0}$  such that  $N = 0 \in \pi_0 \text{Hom}_{\mathcal{C}}(X, X)$ . Therefore, we get a canonical decomposition  $X \simeq \bigvee_{p|N} X_p$  where  $X_p$  is the  $p$ -localization of  $X$ .
- (2) In the above decomposition, for each  $p$  dividing  $N$ ,  $X_p \in \text{Nil}_{\varepsilon, \{p\}}(\mathcal{C})$ .



We will now give basic criteria for identifying  $\varepsilon$ -objects, which shows that they are ‘strongly null-isogenous’ or ‘endomorphism-dissonant.’ Here we will freely use the nilpotence technology of [DHS88, HS98].

Recall that for every  $n$  and prime  $p$ , there exists a type  $n$ , finite  $p$ -local spectrum  $F_n$  admitting a non-trivial  $v_n$ -self map  $f$ . Let  $T(n) = F_n[f^{-1}]$  be the mapping telescope on  $f$ . While the spectrum  $T(n)$  depends on the choice of  $F_n$ , its Bousfield class does not. In other words, the condition  $T(n)_*X = 0$  (for a spectrum  $X$ ) does not depend on the choices made.

**Proposition 2.7.** Let  $\mathcal{C} \in \widehat{\text{Cat}}_\infty^{\text{perf}}$  and  $X \in \mathcal{C}$ . The following are equivalent:

- (1) The object  $X$  is an  $\varepsilon$ -object.
- (2) The endomorphism ring spectrum  $\text{End}_{\mathcal{C}}(X)$  has the property that  $T(n)_*\text{End}_{\mathcal{C}}(X) = 0$  for all  $n \in [0, \infty)$  and all primes  $p$ .
- (3) The endomorphism ring spectrum  $\text{End}_{\mathcal{C}}(X)$  has the property that  $K(n)_*\text{End}_{\mathcal{C}}(X) = 0$  for all  $n \in [0, \infty)$  and all primes  $p$ .

*Proof.* First we prove  $((1) \Rightarrow (2))$ , i.e., if  $X$  is an  $\varepsilon$ -object, then  $T(n)_*\text{End}_{\mathcal{C}}(X) = 0$  for every  $n$  and implicit prime  $p$ . By Proposition 2.6, we can suppose that  $X \in \text{Nil}_{\varepsilon, \{p\}}(\mathcal{C})$ . Now for a fixed  $n \geq 0$ , we choose a type  $n$ , finite  $p$ -local spectrum  $F_n$  with a  $v_n$ -self map  $f$  such that  $T(n) = F_n[f^{-1}]$  as above. Let  $Cf$  denote the cofiber of  $f$ . We will show that the class  $\mathcal{D}$  consisting of all  $Y \in \mathcal{C}$  such that  $T(n)_*\text{Hom}_{\mathcal{C}}(X, Y) = 0$  contains  $X$ . Evidently,  $\mathcal{D}$  is a thick subcategory and by construction of  $Cf$ ,

$$T(n)_*\text{Hom}_{\mathcal{C}}(X, Cf \wedge Z) \simeq T(n)_*(Cf \wedge \text{Hom}_{\mathcal{C}}(X, Z)) = 0, \quad \text{for any } Z \in \mathcal{C},$$

which implies that any  $Cf \wedge Z$  belongs to  $\mathcal{D}$ . By hypothesis,  $X$  belongs to the thick subcategory generated by  $\{Cf \wedge Z\}_{Z \in \mathcal{C}}$  and thus  $X \in \mathcal{D}$ , so the claim follows.

The implication  $((2) \Rightarrow (3))$ , is both easy and classical.

For the final implication  $((3) \Rightarrow (1))$ , suppose that the endomorphism ring spectrum  $\text{End}_{\mathcal{C}}(X)$  is  $K(n)$ -acyclic for all primes  $p$  and  $0 \leq n < \infty$ . First of all, observe that  $\text{End}_{\mathcal{C}}(X)$  is therefore a torsion spectrum. Breaking  $X$  into a direct sum of its  $p$ -localizations, we may assume that  $\text{End}_{\mathcal{C}}(X)$  is  $p$ -power torsion. We then need to show that  $X \in \text{Nil}_{\varepsilon, \{p\}}(\mathcal{C})$ .

Let  $F$  be any finite nontrivial  $p$ -torsion spectrum. We will show that  $X$  belongs to the thick subcategory generated by  $F \wedge X$ . Replacing  $F$  by  $F \wedge \mathbb{D}F = \text{End}_{\text{Sp}}(F)$ , we may assume  $F$  is a ring spectrum and we let  $S^0 \rightarrow F$  be the unit map. We let  $I \rightarrow S^0$  denote the fiber of the unit map. Note that  $F$  has nontrivial homology, so  $I \rightarrow S^0$  induces the zero map in mod  $p$  homology. We will show that there exists  $n$  such that  $I^{\wedge n} \wedge X \rightarrow X$  is nullhomotopic. It will follow that  $X$  is a summand of the cofiber of this map, which belongs to the thick subcategory generated by  $F \wedge X$  as desired.

To show that there exists  $n$  such that  $I^{\wedge n} \wedge X \rightarrow X$  is nullhomotopic, we observe that it suffices to show (by adjointness) that there exists  $n$  such that the map of spectra  $I^{\wedge n} \rightarrow S^0 \rightarrow \text{End}_{\mathcal{C}}(X)$  is nullhomotopic. Equivalently, the map  $S^0 \rightarrow \text{End}_{\mathcal{C}}(X) \wedge \mathbb{D}I$  needs to be *smash nilpotent*. By the nilpotence theorem [HS98, Thm. 3], it suffices to check this after applying  $K(n)_*$  for  $n \in [0, \infty]$ . For  $n < \infty$ , we know that  $\text{End}_{\mathcal{C}}(X)$  itself is  $K(n)_*$ -acyclic. For  $n = \infty$ , so that  $K(\infty) = H\mathbb{F}_p$ , we know that  $S^0 \rightarrow \mathbb{D}I$  induces the zero map in homology, so we are done here too.  $\square$

**Corollary 2.8.** Given  $\mathcal{C} \in \widehat{\text{Cat}}_\infty^{\text{perf}}$  and an object  $X \in \mathcal{C}$ , the following are equivalent:

- (1)  $X$  is an  $\varepsilon$ -object.
- (2) For every prime  $p$  and height  $n \geq 0$  and every exact functor  $F: \mathcal{C} \rightarrow L_{K(n)}\text{Sp}$ , the image of  $X$  under  $F$  is zero.

- (3) For every prime  $p$  and height  $n \geq 0$  and every exact functor  $F: \mathcal{C} \rightarrow L_{T(n)}\mathrm{Sp}$ , the image of  $X$  under  $F$  is zero.

*Proof.* To see that  $((1) \Rightarrow (2) \text{ and } (3))$ , it suffices by Proposition 2.2 to show that there are no nontrivial  $\varepsilon$ -objects of  $L_{K(n)}\mathrm{Sp}$  and  $L_{T(n)}\mathrm{Sp}$ . Let  $C$  be a type  $n+1$  finite complex. Then smashing with  $C$  annihilates both  $L_{K(n)}\mathrm{Sp}$  and  $L_{T(n)}\mathrm{Sp}$ . This easily implies that there are no nontrivial  $\varepsilon$ -objects as desired. For the other direction, i.e., to show  $((2) \text{ or } (3) \Rightarrow (1))$ , we use Proposition 2.7 and consider the functors  $L_{K(n)}\mathrm{Hom}_{\mathcal{C}}(X, \cdot): \mathcal{C} \rightarrow L_{K(n)}\mathrm{Sp}$  (or the telescopic analog).  $\square$

**Remark 2.9.** The results above indicate that our methods do not see any possible distinction between  $T(n)$  and  $K(n)$ -localizations. As a consequence our descent results will be applicable to any of the following localization functors which we will call *periodic* localization functors:  $\{L_{T(n)}, L_{K(n)}, L_n^f, L_n\}_{n \geq 0, p \text{ prime}}$ . Of course, the results for all of the telescopic localizations imply the others.

**Example 2.10.** Here are some examples of  $\varepsilon$ -objects.

- (1) If  $\mathcal{C} = \mathrm{Sp}^\omega$  is the  $\infty$ -category of finite spectra, then every  $\varepsilon$ -object is 0: Using Proposition 2.6 and Proposition 2.7, it suffices to see that if  $X$  is the  $p$ -localization of a finite spectrum with  $K(n)_*(X) = 0$  for all  $0 \leq n < \infty$ , then  $X$  is contractible. Since for sufficiently large  $n$ , the Atiyah-Hirzebruch spectral sequence  $H^*(X, \mathbb{F}_p) \Rightarrow K(n)^*(X)$  collapses, this is clear.
- (2) If  $\mathcal{C}$  is such that for every prime  $p$ , the  $p$ -localization  $\mathcal{C}_{(p)}$  is enriched over  $\mathrm{Mod}(L_n S)$  for some  $n$  (possibly depending on  $p$ ), then every  $\varepsilon$ -object of  $\mathcal{C}$  is zero. This is immediate from Proposition 2.7 and applies for example to the  $E_n$ -local category, the  $K(n)$ -local category, the  $\infty$ -category of  $KO$ -modules, and the  $\infty$ -category of  $Tmf$ -modules.
- (3) If  $\mathcal{C} = \mathrm{Sp}$ , then every  $X \in \mathrm{Sp}$  which admits the structure of an  $H\mathbb{Z}$ -module spectrum and is killed by some  $N > 1$  is an  $\varepsilon$ -object. So is any spectrum in the thick subcategory generated by these, e.g., any spectrum with only finitely many non-zero homotopy groups, each of which is killed by some  $N$ .
- (4) If each mapping spectrum in  $\mathcal{C}$  is an  $H\mathbb{Z}$ -module, for example if  $\mathcal{C}$  arises from a dg-category, then the  $\varepsilon$ -objects are exactly the objects killed by some  $N \in \mathbb{N}$ .
- (5) If  $\mathcal{C}$  is a full subcategory of  $\mathcal{D}$ , then an  $X \in \mathcal{C}$  is an  $\varepsilon$ -object of  $\mathcal{C}$  if and only if it is an  $\varepsilon$ -object of  $\mathcal{D}$ .
- (6)  $H\mathbb{Q}/\mathbb{Z} \in \mathrm{Sp}$  is an example of a spectrum, even an  $H\mathbb{Z}$ -module, which is  $T(n)$ -acyclic for every  $n < \infty$  but is *not* an  $\varepsilon$ -spectrum. This follows because for every  $N$ , multiplication by  $N$  is nonzero on  $H\mathbb{Q}/\mathbb{Z}$ .

**Proposition 2.11.** Suppose  $\mathcal{C} \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{perf}}$ . Given a thick subcategory  $\mathcal{T} \subset \mathcal{C}$ , we have  $(\mathcal{T}_\varepsilon)_\varepsilon = \mathcal{T}_\varepsilon$ . In other words,  $\varepsilon$ -enlargement is an idempotent procedure.

*Proof.* Without loss of generality, we can assume  $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$  by writing  $\mathcal{C}$  as a union of small subcategories. Suppose  $X \in (\mathcal{T}_\varepsilon)_\varepsilon$ . By Proposition 2.5, we need to show that the image of  $X$  in  $\mathcal{C}/\mathcal{T}$  is an  $\varepsilon$ -object. Note that the quotient map carries  $\mathcal{T}_\varepsilon$  into  $\mathrm{Nil}_\varepsilon(\mathcal{C}/\mathcal{T})$  and therefore it carries  $(\mathcal{T}_\varepsilon)_\varepsilon$  into  $(\mathrm{Nil}_\varepsilon(\mathcal{C}/\mathcal{T}))_\varepsilon$  by Proposition 2.2. As a result, we need to show that

$$(\mathrm{Nil}_\varepsilon(\mathcal{C}/\mathcal{T}))_\varepsilon = \mathrm{Nil}_\varepsilon(\mathcal{C}/\mathcal{T})$$

because this will imply that the image of  $X$  in  $\mathcal{C}/\mathcal{T}$  is an  $\varepsilon$ -object.

The upshot of this discussion is that, by passage from  $\mathcal{C}$  to  $\mathcal{C}/\mathcal{T}$ , we can assume  $\mathcal{T} = 0$  to begin with, and we make this assumption. So, assume  $X \in (\mathrm{Nil}_\varepsilon(\mathcal{C}))_\varepsilon$ ; we show that  $X$  is itself an  $\varepsilon$ -object.

By Corollary 2.8, it suffices to show that any exact functor  $F: \mathcal{C} \rightarrow L_{K(n)}\mathrm{Sp}$  annihilates  $X$ . As we saw above,  $F$  annihilates  $\mathrm{Nil}_\varepsilon(\mathcal{C})$  and therefore factors through an exact functor  $\overline{F}: \mathcal{C}/\mathrm{Nil}_\varepsilon(\mathcal{C}) \rightarrow L_{K(n)}\mathrm{Sp}$ . However, our assumption is that the image of  $X$  in  $\mathcal{C}/\mathrm{Nil}_\varepsilon(\mathcal{C})$  is an  $\varepsilon$ -object and therefore  $\overline{F}$  annihilates it.  $\square$

**Remark 2.12.** The collection of  $\varepsilon$ -objects is preserved even by arbitrary *additive* functors between idempotent-complete, stable  $\infty$ -categories. Compare [GGN15, Sec. 2] for a treatment of additive  $\infty$ -categories. In fact, if  $F: \mathcal{C} \rightarrow \mathcal{D}$  is an additive functor, then for each  $X \in \mathcal{C}$ , we obtain a natural map of spectra

$$\tau_{\geq 0}\mathrm{Hom}_{\mathcal{C}}(X, X) \rightarrow \tau_{\geq 0}\mathrm{Hom}_{\mathcal{D}}(F(X), F(X)),$$

which carries the unit in  $\pi_0$  to the unit. Using Proposition 2.7 (and the fact that smashing with  $K(n)$  annihilates bounded-above spectra for  $n > 0$ ) one sees easily that  $F(X)$  is also an  $\varepsilon$ -object.

We next check that  $\varepsilon$ -enlargements are invisible to all finite chromatic localizations. We recall briefly the theory of *finite localizations* (cf. [Mil92]). Let  $\mathcal{C}$  be a presentable stable  $\infty$ -category. Fix a prime  $p$  and a height  $n$ . Then we define a localization functor  $L_n^f: \mathcal{C} \rightarrow \mathcal{C}$  as follows. When  $n = 0$ , then  $L_n^f$  is simply rationalization. When  $n > 0$ , choose a nontrivial type  $n+1$ , finite complex  $F$ . Consider the subcategory of all  $X \in \mathcal{C}$  such that  $F \wedge X$  is contractible. This subcategory is closed under all limits and colimits and is the image of a colimit-preserving, idempotent functor  $L_n^f: \mathcal{C} \rightarrow \mathcal{C}$ . We note that  $L_n^f$  annihilates any object in  $\mathcal{C}$  of the form  $F \wedge X$ , and that as a result  $L_n^f$  annihilates  $\mathrm{Nil}_\varepsilon(\mathcal{C})$ . More generally, we have the following:

**Proposition 2.13.** Let  $\mathcal{C}$  be a presentable, stable  $\infty$ -category and let  $\mathcal{T} \subset \mathcal{C}$  be a thick subcategory. For any prime  $p$  and height  $n$ , we have  $L_n^f\mathcal{T} = L_n^f\mathcal{T}_\varepsilon$ , i.e.,  $\mathcal{T}$  and  $\mathcal{T}_\varepsilon$  have the same image under the finite localization functor  $L_n^f$ .

*Proof.* We have an inclusion  $L_n^f\mathcal{T} \subset L_n^f\mathcal{T}_\varepsilon$ , so it suffices to show the other inclusion. Let  $\Sigma$  be any finite set of prime numbers and let  $F$  be a finite torsion spectrum such that  $F_{(q)} \neq 0$  for any  $q \in \Sigma$  and such that  $L_n^f F = 0$ . Then, it follows that  $\mathcal{T}_{\varepsilon, \Sigma} \subset \mathcal{T}_F$  where we use the notation of Definition 2.1. One sees now that  $L_n^f\mathcal{T}_F = L_n^f\mathcal{T}$ , so we are done.  $\square$

We now consider the case where  $\mathcal{C} \in \widehat{\mathrm{CAlg}}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{perf}})$ . That is,  $(\mathcal{C}, \otimes, \mathbf{1})$  is a not necessarily small, symmetric monoidal, idempotent-complete stable  $\infty$ -category, where the tensor structure  $\otimes$  is exact in each variable. Recall that a thick subcategory  $\mathcal{I} \subset \mathcal{C}$  is called a *thick  $\otimes$ -ideal* if for each  $X \in \mathcal{I}$  and  $Y \in \mathcal{C}$ , we have  $X \otimes Y \in \mathcal{I}$ .

**Proposition 2.14.** If  $\mathcal{I} \subset \mathcal{C}$  is a thick  $\otimes$ -ideal, then so is its  $\varepsilon$ -enlargement  $\mathcal{I}_\varepsilon \subset \mathcal{C}$ .

*Proof.* We need to show that if  $Y \in \mathcal{C}$ , then the exact functor  $(-) \otimes Y: \mathcal{C} \rightarrow \mathcal{C}$  preserves  $\mathcal{I}_\varepsilon$ . This follows from Proposition 2.2 because the functor preserves  $\mathcal{I}$ .  $\square$

**2.2.  $\varepsilon$ -nilpotent towers.** We recall some basic definitions when working with *towers* of objects. We refer to [HPS99] or [Mat15b, §3] for more details.

**Definition 2.15.** Suppose  $\mathcal{C} \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{perf}}$ . Let  $\mathrm{Tow}(\mathcal{C})$  denote the  $\infty$ -category  $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathcal{C})$  of *towers* in  $\mathcal{C}$ , and observe that  $\mathrm{Tow}(\mathcal{C}) \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{perf}}$ . We will often abuse notation and denote a tower by  $\{X_i\}$ , the maps  $X_{i+1} \rightarrow X_i$  being understood.

We recall the definitions of two important thick subcategories of  $\mathrm{Tow}(\mathcal{C})$ :

- (1) We let  $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C}) \subset \mathrm{Tow}(\mathcal{C})$  denote the full subcategory spanned by those towers  $\{X_i\}$  such that there exists  $N \in \mathbb{Z}_{\geq 0}$  such that the maps  $X_{i+N} \rightarrow X_i$  are nullhomotopic for all  $i \in \mathbb{Z}_{\geq 0}$ . We will call such towers *nilpotent*.
- (2) We let  $\mathrm{Tow}^{\mathrm{const}}(\mathcal{C})$  denote the thick subcategory of  $\mathrm{Tow}(\mathcal{C})$  generated by  $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C})$  and the constant towers. We will call such towers *quickly converging*.

We will now consider Definition 2.1 for these thick subcategories of the  $\infty$ -category of towers.

**Definition 2.16.** Suppose  $\mathcal{C} \in \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{perf}}$ .

- (1) We define  $\mathrm{Tow}^{\varepsilon, \mathrm{nil}}(\mathcal{C})$  via

$$\mathrm{Tow}^{\varepsilon, \mathrm{nil}}(\mathcal{C}) = \left( \mathrm{Tow}^{\mathrm{nil}}(\mathcal{C}) \right)_{\varepsilon}.$$

We will say that a tower is  $\varepsilon$ -*nilpotent* if it belongs to  $\mathrm{Tow}^{\varepsilon, \mathrm{nil}}(\mathcal{C})$ .

- (2) Let  $\{X_i\}$  be a tower in  $\mathcal{C}$ . Suppose given an object  $X \in \mathcal{C}$  and a map of towers  $\{X\} \rightarrow \{X_i\}$  (if  $\mathcal{C}$  admits inverse limits, then this is equivalent to giving a map  $X \rightarrow \varprojlim X_i$ ). We will say that this map exhibits  $X$  as an  $\varepsilon$ -*nilpotent limit* of the tower  $\{X_i\}$  if the cofiber tower  $\{X_i/X\}$  belongs to  $\mathrm{Tow}^{\varepsilon, \mathrm{nil}}(\mathcal{C})$ .
- (3) Let  $X^{\bullet} \in \mathrm{Fun}(\Delta^+, \mathcal{C})$  be an augmented cosimplicial object of  $\mathcal{C}$ . We say that it is an  $\varepsilon$ -*nilpotent limit diagram* if the natural map of towers  $\{X^{-1}\} \rightarrow \{\mathrm{Tot}_i(X^{\bullet})\}$  exhibits  $X^{-1}$  as an  $\varepsilon$ -nilpotent limit of the target.

**Proposition 2.17.** The  $\varepsilon$ -nilpotent limit diagrams form a thick subcategory of  $\mathrm{Fun}(\Delta^+, \mathcal{C})$ .

*Proof.* Consider the functor  $F: \mathrm{Fun}(\Delta^+, \mathcal{C}) \rightarrow \mathrm{Tow}(\mathcal{C})$  which to  $X^{\bullet}$  associates the cofiber tower  $\{\mathrm{Tot}_i(X^{\bullet})/X^{-1}\}$ . This is an exact functor, and by definition, the  $\varepsilon$ -limit diagrams are exactly those objects such that this cofiber tower belongs to  $\mathrm{Tow}^{\varepsilon, \mathrm{nil}}(\mathcal{C})$ ; in particular, they form a thick subcategory.  $\square$

Using Proposition 2.2, we easily obtain the functoriality of  $\varepsilon$ -nilpotent towers.

**Proposition 2.18.** Let  $\mathcal{C}, \mathcal{D} \in \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{perf}}$  and let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor. Clearly,  $F$  induces an exact functor  $F_*: \mathrm{Tow}(\mathcal{C}) \rightarrow \mathrm{Tow}(\mathcal{D})$ . Then:

- (1)  $F_*(\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C})) \subset \mathrm{Tow}^{\mathrm{nil}}(\mathcal{D})$ .
- (2)  $F_*(\mathrm{Tow}^{\varepsilon, \mathrm{nil}}(\mathcal{C})) \subset \mathrm{Tow}^{\varepsilon, \mathrm{nil}}(\mathcal{D})$ .
- (3) Suppose  $\{X_i\}$  is a tower in  $\mathcal{C}$  and  $\{X\} \rightarrow \{X_i\}$  is a map of towers. Suppose that this map exhibits  $X$  as an  $\varepsilon$ -nilpotent limit of  $\{X_i\}$ . Then the induced map  $\{F(X)\} \rightarrow \{F(X_i)\}$  exhibits  $F(X)$  as an  $\varepsilon$ -nilpotent limit of  $\{F(X_i)\}$ .
- (4) Let  $X^{\bullet} \in \mathrm{Fun}(\Delta^+, \mathcal{C})$  be an augmented cosimplicial object. Suppose  $X^{\bullet}$  is an  $\varepsilon$ -nilpotent limit. Then the augmented cosimplicial object  $F(X^{\bullet}) \in \mathrm{Fun}(\Delta^+, \mathcal{D})$  is an  $\varepsilon$ -nilpotent limit.

For our purposes, it will be important to know that  $\varepsilon$ -nilpotent limit diagrams turn into actual limit diagrams after periodic localization. We include this result below.

**Proposition 2.19.** Suppose  $\mathcal{C}$  is a presentable stable  $\infty$ -category and  $X^{\bullet} \in \mathrm{Fun}(\Delta^+, \mathcal{C})$  is an  $\varepsilon$ -nilpotent limit diagram. Let  $L_n^f$  be any finite localization functor (associated to a prime  $p$  and a height  $n$ ). Then:

- (1) The map

$$X^{-1} \rightarrow \mathrm{Tot}(X^{\bullet})$$

is an  $\varepsilon$ -equivalence.

- (2) The localized Tot tower  $\{L_n^f \text{Tot}_i(X^\bullet)\}$  is quickly converging.
- (3) The maps

$$L_n^f X^{-1} \rightarrow L_n^f \text{Tot}(X^\bullet) \rightarrow \text{Tot}(L_n^f X^\bullet)$$

are equivalences.

*Proof.* We consider items (1) and (3) first. Setting  $\{Y_i\} := \{\text{Tot}_i(X^\bullet)/X^{-1}\}$ , one reduces to showing that if  $\{Y_i\} \in \text{Tow}^{\varepsilon, \text{nil}}(\mathcal{C})$ , then  $\varprojlim Y_i$  is an  $\varepsilon$ -object and

$$L_n^f \varprojlim Y_i \simeq \varprojlim L_n^f Y_i \simeq 0.$$

In fact, consider the two exact functors

$$F_1, F_2: \text{Tow}(\mathcal{C}) \rightarrow \mathcal{C}, \quad F_1(\{Y_i\}) = L_n^f \varprojlim Y_i, \quad F_2(\{Y_i\}) = \varprojlim L_n^f Y_i.$$

Clearly if  $\{Y_i\} \in \text{Tow}^{\text{nil}}(\mathcal{C})$ , then  $F_1(\{Y_i\}) = F_2(\{Y_i\}) = 0$ . It follows from Proposition 2.2 that if  $\{Y_i\} \in \text{Tow}^{\varepsilon, \text{nil}}(\mathcal{C})$ , then  $F_1(\{Y_i\}), F_2(\{Y_i\})$  are both  $\varepsilon$ -objects of  $\mathcal{C}$ . Since they are both  $L_n^f$ -local, however, it follows that they must be contractible. Similarly, considering the functor  $F_3: \text{Tow}(\mathcal{C}) \rightarrow \mathcal{C}$  given by  $F_3(\{Y_i\}) = \varprojlim Y_i$ , we conclude that  $\varprojlim Y_i$  is an  $\varepsilon$ -object.

To prove item (2), it suffices to show that if  $\{Y_i\} \in \text{Tow}^{\varepsilon, \text{nil}}(\mathcal{C})$ , then the tower  $\{L_n^f Y_i\}$  belongs to  $\text{Tow}^{\text{nil}}(\mathcal{C})$ . This follows from Proposition 2.13, which implies that  $\text{Tow}^{\text{nil}}(\mathcal{C})$  and  $\text{Tow}^{\varepsilon, \text{nil}}(\mathcal{C})$  have the same image under  $L_n^f$ .  $\square$

**2.3.  $(A, \varepsilon)$ -nilpotent objects.** Let  $\mathcal{C} \in \text{CAlg}(\widehat{\text{Cat}}_\infty^{\text{perf}})$  be a symmetric monoidal, stable, idempotent-complete  $\infty$ -category with biexact tensor product and let  $A \in \text{Alg}(\mathcal{C})$ .

**Definition 2.20** (Cf. [Bou79, Def. 3.7]). The subcategory  $\text{Nil}^A \subset \mathcal{C}$  is the thick  $\otimes$ -ideal of  $\mathcal{C}$  generated by  $A$ .

We refer to [MNN15a] for a detailed treatment of the theory of  $A$ -nilpotence and to [Mat16] for some applications to analogs of faithfully flat descent theorems. As before, we can define an  $\varepsilon$ -version of the above following the same pattern.

**Definition 2.21.** We define  $\text{Nil}^{A, \varepsilon} = (\text{Nil}^A)_\varepsilon$  as in Definition 2.1 and we call this the subcategory of  $(A, \varepsilon)$ -nilpotent objects in  $\mathcal{C}$ .

We recall the following result. Although the idea is surely classical, we refer to [MNN15a, Prop. 4.7] for a modern exposition. Consider the augmented cobar construction  $\text{CB}_{\text{aug}}^\bullet(A): \Delta^+ \rightarrow \mathcal{C}$ . The underlying cosimplicial object is the cobar construction  $\text{CB}^\bullet(A) \in \text{Fun}(\Delta, \mathcal{C})$

$$A \rightrightarrows A \otimes A \xrightarrow{\rightarrow} \dots,$$

and the augmentation is from the unit.

**Proposition 2.22.** Let  $\mathcal{C} \in \text{CAlg}(\widehat{\text{Cat}}_\infty^{\text{perf}})$  and let  $A \in \text{Alg}(\mathcal{C})$ . Suppose  $X \in \mathcal{C}$  is  $A$ -nilpotent. Then the augmented cosimplicial object  $\text{CB}_{\text{aug}}^\bullet(A) \otimes X$  is a limit diagram and the associated Tot tower is quickly converging.

As a consequence, we can deduce an  $\varepsilon$ -version of the above.

**Proposition 2.23.** Let  $\mathcal{C} \in \text{CAlg}(\widehat{\text{Cat}}_\infty^{\text{perf}})$  and let  $A \in \text{Alg}(\mathcal{C})$ . If  $X \in \mathcal{C}$  is  $(A, \varepsilon)$ -nilpotent, then the augmented cosimplicial object  $\text{CB}_{\text{aug}}^\bullet(A) \otimes X$  is an  $\varepsilon$ -nilpotent limit diagram.

*Proof.* We have an exact functor  $\mathcal{C} \rightarrow \text{Tow}(\mathcal{C})$  which sends

$$X \mapsto \{\text{cofib}(X \rightarrow \text{Tot}_i(\text{CB}^\bullet(A) \otimes X))\}.$$

This carries  $\text{Nil}^A$  into  $\text{Tow}^{\text{nil}}(\mathcal{C})$  by the previous proposition. Therefore, it carries  $\text{Nil}^{A,\varepsilon}$  into  $\text{Tow}^{\varepsilon,\text{nil}}(\mathcal{C})$  by Proposition 2.2, which completes the proof.  $\square$

### 3. NONCOMMUTATIVE MOTIVES

In this section, we will set up more machinery needed for our descent theorems in algebraic  $K$ -theory. These will take place in an appropriate  $\infty$ -category of non-commutative motives which, following [BGT13], is universal as a target for certain invariants of  $\infty$ -categories with some additional structure.

**3.1. General motives.** We begin by reviewing the point of view on connective algebraic  $K$ -theory described in the papers of Blumberg-Gepner-Tabuada [BGT13, BGT14] and its generalization to the setting of stable  $\infty$ -categories linear over a fixed base, which has been developed by Hoyo-Scherotzke-Sibilla [HSS15]. Our treatment will essentially follow theirs. For our purposes, however, it will be necessary to work with a slight variant since we want to consider invariants of  $\infty$ -categories that do not necessarily commute with filtered colimits, such as topological cyclic homology. This will not change the essential ideas.

As before, we consider the  $\infty$ -category  $\text{Cat}_\infty^{\text{perf}}$  of small, idempotent-complete, stable  $\infty$ -categories and exact functors between them. Using the Lurie tensor product,  $\text{Cat}_\infty^{\text{perf}}$  is a symmetric monoidal  $\infty$ -category. One has a lax symmetric monoidal functor

$$K(-): \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Sp}$$

given by (connective)  $K$ -theory. In the setup of [BGT13, BGT14], one constructs a presentable, stable symmetric monoidal  $\infty$ -category  $\text{Mot}$  of *non-commutative motives*. This receives a symmetric monoidal functor  $\mathcal{U}_{\text{add}}: \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Mot}$ , and given  $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$ , we can identify  $K(\mathcal{C})$  with the mapping spectrum from the unit into the object  $\mathcal{U}_{\text{add}}(\mathcal{C}) \in \text{Mot}$ . The  $\infty$ -category  $\text{Mot}$  satisfies a *universal* property for receiving maps from  $\text{Cat}_\infty^{\text{perf}}$  that satisfy certain properties. We briefly review a general form of their construction (cf. [BGT14, §5]).

Let  $\mathcal{C}$  be a small, pointed, symmetric monoidal  $\infty$ -category and let  $\mathfrak{A} \subset \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$  be a full subcategory. Suppose that:

- (1)  $\mathfrak{A}$  is closed under tensoring with objects in  $\mathcal{C}$ .
- (2) The tensor product of a zero object in  $\mathcal{C}$  with any object is a zero object.

**Definition 3.1.** Let  $\mathcal{D}$  be a presentable, stable  $\infty$ -category. A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is called  $\mathfrak{A}$ -*admissible* if the following hold:

- Let  $*$   $\in \mathcal{C}$  be a zero object. Then  $F(*) \in \mathcal{D}$  is a zero object.
- Any diagram  $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$  that belongs to  $\mathfrak{A}$  is carried by  $F$  to a pushout square in  $\mathcal{D}$ .

We will let  $\text{Fun}^{\text{adm}}(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$  denote the  $\infty$ -category of  $\mathfrak{A}$ -admissible functors from  $\mathcal{C}$  to  $\mathcal{D}$ .

We will now recall the construction of the universal presentable, stable  $\infty$ -category receiving an  $\mathfrak{A}$ -admissible functor from  $\mathcal{C}$ , which we will denote  $\text{Mot}_{\mathcal{C}, \mathfrak{A}}$  below.

**Construction 3.2.** We consider the Yoneda embedding  $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$ . We recall (cf. [Lur16, Cor. 4.8.1.12] and [Gla16]) that the target inherits a symmetric monoidal structure via *Day convolution*. Similarly, the  $\infty$ -category  $\mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$  of presheaves of spectra inherits a bicocontinuous symmetric monoidal structure via the tensor product

$$\mathcal{P}_{\mathrm{Sp}}(\mathcal{C}) \simeq \mathcal{P}(\mathcal{C}) \otimes \mathrm{Sp}.$$

We have a canonical symmetric monoidal functor  $\mathcal{C} \rightarrow \mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$ . We will let the image of  $x \in \mathcal{C}$  in  $\mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$  be denoted  $h_x$ ; these are compact generators of  $\mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$ .

We consider the localizing subcategory  $\mathcal{I} \subset \mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$  generated by the following two sets of objects:

- $h_*$ , where  $*$   $\in \mathcal{C}$  is a zero object.
- For each square

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

which belongs to  $\mathfrak{A}$ , the cofiber of  $h_b \sqcup_{h_a} h_c \rightarrow h_d$ .

Our hypotheses on  $\mathfrak{A}$  and  $\mathcal{C}$  imply that  $\mathcal{I}$  is actually a localizing  $\otimes$ -ideal and that it is compactly generated. We define the symmetric monoidal  $\infty$ -category  $\mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$  of  $(\mathcal{C}, \mathfrak{A})$ -motives as the localization of  $\mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$  at the class of morphisms  $X \rightarrow 0$  where  $X \in \mathcal{I}$ .

We note the following basic properties of  $\mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$ :

- Proposition 3.3.** (1)  $\mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$  is a presentable, symmetric monoidal stable  $\infty$ -category with a bicocontinuous tensor product  $\otimes$ .
- (2) There is a symmetric monoidal,  $\mathfrak{A}$ -admissible functor  $\mathcal{C} \rightarrow \mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$  and its image consists of compact objects of  $\mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$ . In particular, the unit is compact in  $\mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$ .
- (3) Let  $\mathcal{D}$  be any presentable stable  $\infty$ -category. Then there is a natural equivalence

$$\mathrm{Fun}^L(\mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{adm}}(\mathcal{C}, \mathcal{D}).$$

If, moreover,  $\mathcal{D}$  is presentably symmetric monoidal, then symmetric monoidal functors correspond under this equivalence.

*Proof.* The first claim follows from the construction of  $\mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$  as the accessible localization of  $\mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$  at a localizing  $\otimes$ -ideal. For the second claim, we note that the functor  $\mathcal{C} \rightarrow \mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$  is symmetric monoidal and so is the localization functor  $\mathcal{P}_{\mathrm{Sp}}(\mathcal{C}) \rightarrow \mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$ . Therefore, their composite is naturally a symmetric monoidal functor. Now the Yoneda functor  $\mathcal{C} \rightarrow \mathcal{P}_{\mathrm{Sp}}(\mathcal{C})$  takes values in compact objects and, since we are localizing at a class of morphisms generated by maps with compact source and target, the localization  $\mathcal{P}_{\mathrm{Sp}}(\mathcal{C}) \rightarrow \mathrm{Mot}_{\mathcal{C}, \mathfrak{A}}$  preserves compact objects. For the final claim, we have natural equivalences for any presentable stable  $\infty$ -category  $\mathcal{D}$ ,

$$\mathrm{Fun}^L(\mathcal{P}_{\mathrm{Sp}}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Here one sees that the subcategory  $\mathrm{Fun}^{\mathrm{adm}}(\mathcal{C}, \mathcal{D})$  of  $\mathfrak{A}$ -admissible functors corresponds to the subcategory of  $\mathrm{Fun}^L(\mathcal{P}_{\mathrm{Sp}}(\mathcal{C}), \mathcal{D})$  which carries the localizing subcategory  $\mathcal{I}$  into 0. This proves the result.  $\square$

**3.2. The  $\infty$ -categories  $\text{Mot}_\kappa(\mathcal{R})$ .** We consider the symmetric monoidal  $\infty$ -category  $\text{Cat}_\infty^{\text{perf}}$  of small stable, idempotent-complete  $\infty$ -categories with the Lurie tensor product. This is a compactly generated, presentable  $\infty$ -category (cf. [BGT13, Cor. 4.25]) with a bicocontinuous tensor product. We will need to know that the compact objects are closed under the tensor product.

**Lemma 3.4** ([BGT14, Prop. 5.2]). The tensor product of compact objects  $\mathcal{C}, \mathcal{D} \in \text{Cat}_\infty^{\text{perf}}$  is compact.

**Definition 3.5.** Given a small symmetric monoidal stable  $\infty$ -category  $\mathcal{R} \in \text{CAlg}(\text{Cat}_\infty^{\text{perf}})$ , consider its  $\infty$ -category  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$  of modules in  $\text{Cat}_\infty^{\text{perf}}$ , which we call the  $\infty$ -category of (small)  $\mathcal{R}$ -linear  $\infty$ -categories.

By construction,  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$  is a *compactly generated* presentable symmetric monoidal  $\infty$ -category with the relative tensor product  $- \otimes_\mathcal{R} -$  commuting with colimits in each variable. The generators are of the form  $\mathcal{R} \otimes \mathcal{C}$ , where  $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$  is compact. It follows that there are also internal mapping objects  $\text{Map}_\mathcal{R}(-, -)$  in  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$ .

**Example 3.6.** Since the unit  $\mathcal{R} \in \text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$  corepresents the functor which extracts the underlying  $\infty$ -groupoid of an object  $\mathcal{M}$  of  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$ , it follows that the underlying  $\infty$ -groupoid of  $\text{Map}_\mathcal{R}(\mathcal{M}, \mathcal{N})$  identifies with the space of maps  $\mathcal{M} \rightarrow \mathcal{N}$  in  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$ .

**Construction 3.7.** The symmetric monoidal  $\infty$ -category  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$  is closed, so we can regard it as enriched over itself. As a result, we can extract a 2-category from it, as follows. Namely, for  $\mathcal{M}, \mathcal{N} \in \text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$ , we define the category of morphisms from  $\mathcal{M}$  to  $\mathcal{N}$  to be the *homotopy category* of the underlying  $\infty$ -category of the internal mapping object  $\text{Map}_\mathcal{R}(\mathcal{M}, \mathcal{N}) \in \text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$ .

Thus, for a morphism  $f: \mathcal{M} \rightarrow \mathcal{N}$  in  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$ , it makes sense to ask, for example, whether  $f$  is right adjointable.

**Remark 3.8.** The condition that a morphism  $f$  in  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$  be right adjointable is generally stronger than the condition that the underlying functor of  $f$  admit a right adjoint (we need the right adjoint to be  $\mathcal{R}$ -linear), though the conditions are equivalent if every object of  $\mathcal{R}$  is dualizable. To avoid ambiguity, we will use the term  *$\mathcal{R}$ -linear right adjoint* for the notion coming from the 2-category  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$ . For more discussion of this issue, see [Gai12] and [Lur, Rem. D.1.5.3].

Now we recall a class  $\mathfrak{A}$  of objects of  $\text{Fun}(\Delta^1 \times \Delta^1, \text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}}))$  which corresponds to the class of *split-exact sequences* of [BGT13, BGT14], or in other terms to the notion of *semi-orthogonal decomposition* in the theory of triangulated categories (cf. [Lur, §6.2]). Compare [HSS15, Def. 5.3] for a discussion in the  $\mathcal{R}$ -linear setting.

**Definition 3.9.** Let  $\mathcal{R} \in \text{CAlg}(\text{Cat}_\infty^{\text{perf}})$  and let

$$\mathcal{X} = (\mathcal{M} \xrightarrow{i} \mathcal{N} \xrightarrow{p} \mathcal{P})$$

a sequence in  $\text{Mod}_\mathcal{R}(\text{Cat}_\infty^{\text{perf}})$  with null composite. Note that the space of null-homotopies of any map in  $\text{Cat}_\infty^{\text{perf}}$  is either empty or contractible, so this can be interpreted either as structure or a condition.

We say that  $\mathcal{X}$  is a *split-exact sequence* if the following conditions hold:

- The functors  $i$  and  $p$  have  $\mathcal{R}$ -linear both have right adjoints, say  $i_r$  and  $p_r$ , respectively.
- The unit map  $1 \rightarrow i_r \circ i$  is an equivalence, i.e.,  $i$  is fully faithful.
- The counit map  $p \circ p_r \rightarrow 1$  is an equivalence, i.e.,  $p_r$  is fully faithful.



- The natural sequence  $i \circ i_r \rightarrow \text{id}_{\mathcal{N}} \rightarrow p_r \circ p$  is a cofiber sequence of functors  $\mathcal{N} \rightarrow \mathcal{N}$ . Note that one has a canonical nullhomotopy of the composite because  $\mathcal{M} \rightarrow \mathcal{P}$  is the zero functor.

In such a situation, it is easy to see that  $\mathcal{X}$  is both a fiber sequence and a cofiber sequence in  $\text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})$  (and a Verdier quotient in  $\text{Cat}_{\infty}^{\text{perf}}$ ), and the same is true of the sequence obtained by passing to right adjoints. We will need to know that the  $\mathcal{R}$ -linear tensor product respects split-exact sequences. Compare also [BGT14, Lem. 5.5].

**Lemma 3.10.** Let  $\mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{P}$  be a split-exact sequence of  $\mathcal{R}$ -linear  $\infty$ -categories and let  $\mathcal{C}$  be any  $\mathcal{R}$ -linear  $\infty$ -category. Then the sequence  $\mathcal{C} \otimes_{\mathcal{R}} \mathcal{M} \rightarrow \mathcal{C} \otimes_{\mathcal{R}} \mathcal{N} \rightarrow \mathcal{C} \otimes_{\mathcal{R}} \mathcal{P}$  is a split-exact sequence too.

*Proof.* The tensor product  $-\otimes_{\mathcal{R}} \mathcal{C} : \text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}}) \rightarrow \text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})$  is naturally an enriched functor. Therefore, it can be applied not only to  $\mathcal{R}$ -linear functors, but also to natural transformations of functors: it has the structure of a 2-functor. In particular, it preserves adjunctions, and hence it preserves the first three conditions in the definition of split-exact sequence.

To check the last condition, we observe that if  $F' \rightarrow F \rightarrow F''$  is a cofiber sequence of  $\mathcal{R}$ -linear functors  $\mathcal{C} \rightarrow \mathcal{D}$  for  $\mathcal{C}, \mathcal{D} \in \text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})$ , then

$$F' \otimes_{\mathcal{R}} \mathcal{E} \rightarrow F \otimes_{\mathcal{R}} \mathcal{E} \rightarrow F'' \otimes_{\mathcal{R}} \mathcal{E}$$

is a cofiber sequence of  $\mathcal{R}$ -linear functors  $\mathcal{C} \otimes_{\mathcal{R}} \mathcal{E} \rightarrow \mathcal{D} \otimes_{\mathcal{R}} \mathcal{E}$  for any  $\mathcal{E} \in \text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})$ . This follows from the fact that the map of spaces

$$\text{Hom}_{\text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Hom}_{\text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})}(\mathcal{C} \otimes_{\mathcal{R}} \mathcal{E}, \mathcal{D} \otimes_{\mathcal{R}} \mathcal{E})$$

arises by taking spaces of objects from a morphism

$$\text{Map}_{\mathcal{R}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Map}_{\mathcal{R}}(\mathcal{C} \otimes_{\mathcal{R}} \mathcal{E}, \mathcal{D} \otimes_{\mathcal{R}} \mathcal{E}) \in \text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}}),$$

i.e., an  $\mathcal{R}$ -linear exact functor, which in particular preserves cofiber sequences on underlying  $\infty$ -categories.  $\square$

We now would like to construct an  $\infty$ -category of  $\mathcal{R}$ -linear non-commutative motives from  $\text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})$ , but we need to make a minor technical detour since  $\text{Cat}_{\infty}^{\text{perf}}$  is not essentially small. Choose a regular cardinal  $\kappa$ . Then the  $\kappa$ -compact objects in  $\text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})$  are closed under pushouts, retracts, and  $\mathcal{R}$ -linear tensor products (cf. Lemma 3.4). For any specific statement (e.g., descent-theoretic assertion), we will end up assuming  $\kappa$  is taken large enough such that our required statement takes place entirely in the world of  $\kappa$ -compact objects.

**Definition 3.11.** Fix  $\mathcal{R} \in \text{CAlg}(\text{Cat}_{\infty}^{\text{perf}})$  and fix a regular cardinal  $\kappa$ . We let  $\text{Mod}_{\mathcal{R}}^{\kappa}(\text{Cat}_{\infty}^{\text{perf}}) \subset \text{Mod}_{\mathcal{R}}(\text{Cat}_{\infty}^{\text{perf}})$  denote the full subcategory spanned by the  $\kappa$ -compact objects, or an equivalent small model.

- (1) The  $\infty$ -category  $\text{Mot}_{\kappa}(\mathcal{R})$  of *non-commutative  $\mathcal{R}$ -motives* is defined to be  $\text{Mot}_{\text{Mod}_{\mathcal{R}}^{\kappa}(\text{Cat}_{\infty}^{\text{perf}}), \mathfrak{A}}$  where  $\mathfrak{A}$  is the collection of split-exact sequences in  $\text{Mod}_{\mathcal{R}}^{\kappa}(\text{Cat}_{\infty}^{\text{perf}})$ . When  $\mathcal{R} = \text{Sp}^{\omega}$  and  $\kappa = \aleph_0$ , then this is equivalent to the definition of [BGT13] in view of [HSS15, Prop. 5.6]. Of course, the definition of  $\text{Mot}_{\kappa}(\mathcal{R})$  depends on the choice of  $\kappa$ .
- (2) When  $R$  is an  $\mathbb{E}_{\infty}$ -ring, we will write  $\text{Mot}_{\kappa}(R) = \text{Mot}_{\kappa}(\text{Perf}(R))$ .
- (3) Given a  $\kappa$ -compact  $\mathcal{R}$ -linear  $\infty$ -category  $\mathcal{M}$ , we write  $\mathcal{U}_{\text{wadd}}(\mathcal{M}) \in \text{Mot}_{\kappa}(\mathcal{R})$  for its image. We let  $K'_0(\mathcal{M}) = \pi_0 \text{Hom}_{\text{Mot}_{\kappa}(\mathcal{R})}(\mathcal{U}_{\text{wadd}}(\mathcal{R}), \mathcal{U}_{\text{wadd}}(\mathcal{M}))$ .

- (4) A *weakly additive invariant* of ( $\kappa$ -compact)  $\mathcal{R}$ -linear  $\infty$ -categories is a functor  $\mathrm{Mod}_{\mathcal{R}}^{\kappa}(\mathrm{Cat}_{\infty}^{\mathrm{perf}}) \rightarrow \mathcal{D}$ , where  $\mathcal{D}$  is a presentable stable  $\infty$ -category, which is  $\mathfrak{A}$ -admissible for  $\mathfrak{A}$  as in part (1). We find that any weakly additive invariant naturally factors through  $\mathrm{Mot}_{\kappa}(\mathcal{R})$  by Proposition 3.3. We adopt the term *weakly additive* here to remind the reader that commutation with filtered colimits is not stipulated.

**Construction 3.12.** Let  $\mathcal{M}$  be an  $\mathcal{R}$ -linear  $\infty$ -category. Let  $Ex(\mathcal{M})$  denote the  $\mathcal{R}$ -linear  $\infty$ -category of cofiber sequences  $(X \rightarrow Y \rightarrow Z)$  in  $\mathcal{M}$ . More formally,  $Ex(\mathcal{M}) \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M})$  is the subcategory of those cocartesian diagrams of the following form:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

Note that  $Ex(\mathcal{M}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{M})$  via  $(X \rightarrow Y \rightarrow Z) \mapsto (X \rightarrow Y)$ . We have the following split-exact sequence of  $\mathcal{R}$ -linear  $\infty$ -categories  $\mathcal{M} \rightarrow Ex(\mathcal{M}) \rightarrow \mathcal{M}$ , where:

- $\mathcal{M} \rightarrow Ex(\mathcal{M})$  is the functor  $X \mapsto (X = X \rightarrow 0)$ . This has a right adjoint which sends  $(X \rightarrow Y \rightarrow Z)$  to  $X$ .
- $Ex(\mathcal{M}) \rightarrow \mathcal{M}$  is the functor  $(X \rightarrow Y \rightarrow Z) \mapsto Z$ . This has a right adjoint which sends  $Z$  to  $(0 \rightarrow Z = Z)$ .

**Remark 3.13.** Note that  $Ex(\mathcal{M}) \simeq Ex(\mathrm{Sp}^{\omega}) \otimes \mathcal{M} \in \mathrm{Cat}_{\infty}^{\mathrm{perf}}$ , so that if  $\mathcal{M} \in \mathrm{Mod}_{\mathcal{R}}(\mathrm{Cat}_{\infty}^{\mathrm{perf}})$  is  $\kappa$ -compact, then so is  $Ex(\mathcal{M})$ . To see this, we use the fact that  $Ex(\mathcal{M}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{M})$  is the  $\infty$ -category of compact objects in  $\mathrm{Fun}(\Delta^1, \mathrm{Ind}(\mathcal{M})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Sp}) \otimes \mathrm{Ind}(\mathcal{M})$ , where the latter tensor product is taken in the  $\infty$ -category  $\mathcal{P}r_{\mathrm{st}}^L$ . Compare [Lur15, Prop. 2.2.6].

It follows that the functors  $Ex(\mathcal{M}) \rightrightarrows \mathcal{M}$  given by  $(X \rightarrow Y \rightarrow Z) \mapsto X$  and  $(X \rightarrow Y \rightarrow Z) \mapsto Z$  induce an equivalence

$$\mathcal{U}_{\mathrm{wadd}}(Ex(\mathcal{M})) \simeq \mathcal{U}_{\mathrm{wadd}}(\mathcal{M}) \times \mathcal{U}_{\mathrm{wadd}}(\mathcal{M}) \in \mathrm{Mot}_{\kappa}(\mathcal{R}).$$

**Construction 3.14.** Let  $\mathcal{M}$  be a  $\kappa$ -compact  $\mathcal{R}$ -linear  $\infty$ -category. We construct a homomorphism

$$(3.15) \quad K_0(\mathcal{M}) \rightarrow K'_0(\mathcal{M})$$

as follows.

Given  $X \in \mathcal{M}$ , we obtain an  $\mathcal{R}$ -linear functor  $\mathcal{R} \rightarrow \mathcal{M}$  sending the unit object to  $X$ , which defines an element in  $K'_0(\mathcal{M})$ . Given a cofiber sequence  $X \rightarrow Y \rightarrow Z$ , we need to show that the class of  $Y$  in  $K'_0(\mathcal{M})$  is equal to the sum of the classes of  $X$  and  $Z$ . By applying the functor of projection to the ‘middle term,’ it suffices to check that the class of the object  $(X \rightarrow Y \rightarrow Z)$  in  $Ex(\mathcal{M})$  is the sum of the classes  $(X = X \rightarrow 0)$  and  $(0 \rightarrow Z = Z)$ . But by the above equivalence in  $\mathrm{Mot}_{\kappa}(\mathcal{R})$ , we can check this after projection to the outer terms, where it is obvious.

Using the techniques of [BGT13, HSS15], one can in fact show that  $K_0 \simeq K'_0$ . Since we will not need this, we omit the proof.

#### 4. ABSTRACT DESCENT RESULTS

In this brief but central section, we describe how the use of  $\mathbb{E}_{\infty}$ -structures enables one to prove abstract descent and  $\varepsilon$ -nilpotence results in a symmetric monoidal, stable  $\infty$ -category. Our basic tool is the following (Theorem 4.2). Throughout, we use the following notation.

**Definition 4.1.** Given  $\mathcal{C} \in \text{CAlg}(\widehat{\text{Cat}}_\infty^{\text{perf}})$  and  $M \in \mathcal{C}$ , we let  $\langle M \rangle^\otimes$  denote the thick  $\otimes$ -ideal generated by  $M$  and  $\langle M \rangle_\varepsilon^\otimes$  its  $\varepsilon$ -enlargement.

**Theorem 4.2.** Suppose  $\mathcal{C} \in \text{CAlg}(\widehat{\text{Cat}}_\infty^{\text{perf}})$  with unit  $\mathbf{1}$  and  $R \in \text{CAlg}(\mathcal{C})$ . Moreover, suppose there exists  $M \in \mathcal{C}$  and a map  $M \rightarrow R$  such that the image of  $(\pi_0 M) \otimes \mathbb{Q} \rightarrow (\pi_0 R) \otimes \mathbb{Q}$  contains the unit. Then  $R \in \langle M \rangle_\varepsilon^\otimes$ .

*Proof.* Without loss of generality, we may assume  $\mathcal{C}$  is small by writing  $\mathcal{C}$  as a union of small subcategories. Let  $N$  be a positive integer such that  $(\pi_0 M)[N^{-1}] \rightarrow (\pi_0 R)[N^{-1}]$  has image containing the unit. Let  $\Sigma$  be the set of primes dividing  $N$ . We will show that in fact  $R$  belongs to  $\langle M \rangle_{\varepsilon, \Sigma}^\otimes$ .

To see this, let  $T$  be any finite spectrum whose  $p$ -localizations for  $p \in \Sigma$  are all nontrivial. We need to show that  $R$  belongs to the thick  $\otimes$ -ideal generated by  $M$  and  $T \wedge \mathbf{1}$ . Let  $\mathcal{J} \subset \mathcal{C}$  denote this thick  $\otimes$ -ideal. We can then form the Verdier quotient  $\mathcal{C}/\mathcal{J}$ , and equivalently we need to show that the image  $\overline{R}$  of  $R$  in  $\mathcal{C}/\mathcal{J}$  is zero. This is equivalent to showing that the  $\mathbb{E}_\infty$ -ring  $B = \text{Hom}_{\mathcal{C}/\mathcal{J}}(\mathbf{1}, \overline{R})$  is contractible: in fact, that will imply that the unit map  $\mathbf{1} \rightarrow \overline{R}$  is nullhomotopic, so that  $\overline{R} = 0$ . Note also that  $T \wedge B = 0$  since smashing with  $T$  annihilates the  $\infty$ -category  $\mathcal{C}/\mathcal{J}$ .

We now prove that  $B$  is contractible. By hypothesis, there is a map  $f: M \rightarrow R$  in  $\mathcal{C}$  whose image in  $\pi_0 \otimes \mathbb{Z}[N^{-1}]$  contains the unit. In other words, there is a map  $g: \mathbf{1} \rightarrow M$  such that the composite  $x := f \circ g \in \pi_0 R$  satisfies  $N^k(x - 1) = 0$  for some  $k \geq 0$ . After taking the Verdier quotient,  $M$  and hence  $x$  maps to zero, so we find that  $N^k = 0$  in  $\pi_0 B$ . This implies that we have  $H\mathbb{Q} \wedge B = 0$  and that for all primes  $p$  not dividing  $N$ , we also have  $H\mathbb{F}_p \wedge B = 0$ . For each  $p \mid N$ , however, we claim that we have  $H\mathbb{F}_p \wedge B = 0$  as well. This follows from the fact that  $B \wedge T = 0$  and since  $T$  has nontrivial mod  $p$  homology. Using the May nilpotence conjecture [MNN15b, Thm. A] applied to the unit in  $B$ , we conclude that  $B = 0$ .  $\square$

We obtain the following consequence for  $(A, \varepsilon)$ -nilpotence.

**Theorem 4.3.** Suppose  $\mathcal{C} \in \text{CAlg}(\widehat{\text{Cat}}_\infty^{\text{perf}})$ ,  $R \in \text{CAlg}(\mathcal{C})$  and  $A \in \text{Alg}(\mathcal{C})$ . Suppose that there exists an  $A$ -module  $M$  and a map  $M \rightarrow R$  such that the image of  $\pi_0 M \otimes \mathbb{Q} \rightarrow \pi_0 R \otimes \mathbb{Q}$  contains the unit. Then  $R \in \text{Nil}^{A, \varepsilon}$ . In particular,  $\text{CB}_{\text{aug}}^\bullet(A) \otimes R$  is an  $\varepsilon$ -nilpotent limit diagram.

*Proof.* Theorem 4.2 implies that  $R \in \langle M \rangle_\varepsilon^\otimes$ . Since  $M$  is an  $A$ -module, we have  $M \in \langle A \rangle^\otimes$ , and conclude that  $R \in \langle A \rangle_\varepsilon^\otimes = \text{Nil}^{A, \varepsilon}$ . The final claim follows from Proposition 2.23.  $\square$

We can apply this to obtain an  $\varepsilon$ -nilpotence descent result in  $\text{Mot}_\kappa(\mathcal{R})$  in the sense of the previous section.

**Theorem 4.4.** Let  $\mathcal{R} \in \text{CAlg}(\text{Cat}_\infty^{\text{perf}})$  be a small symmetric monoidal, idempotent-complete stable  $\infty$ -category and let  $\mathcal{A} \in \text{Alg}(\text{Mod}_{\mathcal{R}}(\text{Cat}_\infty^{\text{perf}}))$  be an  $\mathcal{R}$ -linear monoidal  $\infty$ -category. Suppose that there exists an  $\mathcal{R}$ -linear functor  $\mathcal{A} \rightarrow \mathcal{R}$  whose image on  $K_0(-) \otimes \mathbb{Q}$  contains the unit. Choose a regular cardinal  $\kappa$  such that  $\mathcal{A}$  is  $\kappa$ -compact in  $\text{Mod}_{\mathcal{R}}(\text{Cat}_\infty^{\text{perf}})$ . Then the augmented cosimplicial object in  $\text{Mot}_\kappa(\mathcal{R})$ ,

$$\mathcal{U}_{\text{wadd}}(\mathcal{R}) \rightarrow \left( \mathcal{U}_{\text{wadd}}(\mathcal{A}) \rightrightarrows \mathcal{U}_{\text{wadd}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}) \xrightarrow{\quad} \dots \right)$$

is an  $\varepsilon$ -nilpotent limit.

*Proof.* We apply Theorem 4.3 with  $\mathcal{C} := \text{Mot}_\kappa(\mathcal{R})$ ,  $R := \mathcal{U}_{\text{wadd}}(\mathcal{R}) \in \text{CAlg}(\mathcal{C})$  the unit of  $\mathcal{C}$ , and  $M := \mathcal{A} := \mathcal{U}_{\text{wadd}}(\mathcal{A}) \in \text{Alg}(\mathcal{C})$ . Then the map  $(\pi_0 M \rightarrow \pi_0 R) = (K'_0(\mathcal{A}) \rightarrow K'_0(\mathcal{R}))$  is seen to be rationally surjective (equivalently, has image containing the unit) using the map in 3.14 and our assumption on  $K_0$ .  $\square$

To conclude this section, we now establish a result that gives a much stronger conclusion about the comparison map as above in the special case in which one has a further assumption on the image in  $\pi_0$  of the map  $M \rightarrow R$ .

**Theorem 4.5.** Suppose  $\mathcal{C}$  is a presentable, symmetric monoidal stable  $\infty$ -category where the tensor is bicontinuous, and the unit  $\mathbf{1}$  is compact. Suppose  $A \in \text{Alg}(\mathcal{C})$  is dualizable in  $\mathcal{C}$ . Suppose that there exists an  $A$ -module  $M$  and a map  $M \rightarrow \mathbf{1}$  such that the image of  $\pi_0 M \rightarrow \pi_0 \mathbf{1}$  contains the prime  $p$ . Then for any  $X \in \mathcal{C}$ , the fiber of

$$X \rightarrow \text{Tot}(\text{CB}^\bullet(A) \otimes X)$$

has the structure of an  $\mathbb{F}_p$ -module.

*Proof.* We will freely use the language of acyclizations (or cellularizations) and localizations with respect to a dualizable object, for which we refer to [MNN15a, § 2–3] for an exposition. We use  $\mathbf{Hom}$  for internal mapping objects in  $\mathcal{C}$ . Let  $V_A, U_A \in \mathcal{C}$  be the  $A$ -acyclization and  $A^{-1}$ -localization of  $\mathbf{1}$ , so that we have a cofiber sequence

$$V_A \rightarrow \mathbf{1} \rightarrow U_A,$$

such that  $A \otimes U_A = 0$  and such that  $V_A$  belongs to the localizing  $\otimes$ -ideal generated by  $A$ . We then have

$$\text{Tot}(\text{CB}^\bullet(A) \otimes X) \simeq \mathbf{Hom}_{\mathcal{C}}(V_A, X),$$

because both sides give the  $A$ -completion of  $X$ . In particular, the fiber of  $X \rightarrow \text{Tot}(\text{CB}^\bullet(A) \otimes X)$  can be identified with the internal mapping object  $\mathbf{Hom}_{\mathcal{C}}(U_A, M)$ .

We observe now that  $U_A$  is an  $\mathbb{E}_\infty$ -algebra in  $\mathcal{C}$ , as the  $A^{-1}$ -localization of  $\mathbf{1}$ . Moreover, by assumption  $\pi_0 U_A$  is an  $\mathbb{F}_p$ -algebra. It follows that the underlying  $\mathbb{E}_2$ -algebra of  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, U_A)$  is an  $\mathbb{E}_2$ -algebra over  $\mathbb{F}_p$  by the Hopkins-Mahowald theorem that the free  $\mathbb{E}_2$ -algebra with  $p = 0$  is  $H\mathbb{F}_p$ . See [MNN15b, Thm. 4.18] or [ACB14, Thm. 5.1] for recent expositions of this result. By adjunction, it follows that  $U_A \in \mathcal{C}$  is a module over  $\mathbb{F}_p$ , and therefore the fiber  $\mathbf{Hom}_{\mathcal{C}}(U_A, X)$  is one, too.  $\square$

## 5. EXAMPLES AND APPLICATIONS

In this section, we will give the primary examples and applications of our descent results (in particular, Theorem 4.4), to algebraic finite flat extensions of  $\mathbb{E}_\infty$ -rings and to many of Rognes-style Galois extensions of  $\mathbb{E}_\infty$ -rings.

**Theorem 5.1.** Let  $A$  be an  $\mathbb{E}_\infty$ -ring. Let  $B$  be an  $\mathbb{E}_2$ -algebra in the  $\infty$ -category of  $A$ -modules. Suppose that  $B$  is a perfect  $A$ -module and the natural map  $K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$  has image containing the unit. Assume  $\kappa$  is such that  $\text{Perf}(B)$  is  $\kappa$ -compact in  $\text{Mod}_{\text{Perf}(A)}(\text{Cat}_\infty^{\text{perf}})$ .

Let  $F$  be any weakly additive invariant of  $\kappa$ -compact  $A$ -linear  $\infty$ -categories (e.g.,  $K$ -theory, non-connective/Bass  $K$ -theory, homotopy  $K$ -theory of  $\mathbb{Z}$ -linear  $\infty$ -categories [Tab15],  $THH$ , and  $TC$ ) taking values in a presentable stable  $\infty$ -category  $\mathcal{D}$ . Then the augmented cosimplicial diagram

$$F(\text{Perf}(A)) \rightarrow \left( F(\text{Perf}(B)) \rightrightarrows F(\text{Perf}(B \otimes_A B)) \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} \dots \right)$$

is an  $\varepsilon$ -nilpotent limit. In particular, the natural map in  $\mathcal{D}$

$$F(\text{Perf}(A)) \rightarrow \text{Tot} \left( F(\text{Perf}(B)) \rightrightarrows F(\text{Perf}(B \otimes_A B)) \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} \dots \right)$$

is an  $\varepsilon$ -equivalence, and the associated Tot tower becomes quickly convergent after any periodic localization.

*Proof.* We apply Theorem 4.4 with  $\mathcal{R} := \text{Perf}(A)$ ,  $\mathcal{A} := \text{Perf}(B)$  (which is monoidal as  $B$  is an  $\mathbb{E}_2$ -algebra) and the forgetful functor  $\mathcal{A} \rightarrow \mathcal{R}$ , which is  $\mathcal{R}$ -linear. Note that the functor  $B' \mapsto \text{Perf}(B')$  is a symmetric monoidal functor from the  $\infty$ -category of  $A$ -algebras to the  $\infty$ -category  $\text{Mod}_{\text{Perf}(A)}(\text{Cat}_{\infty}^{\text{perf}})$  (cf. [Lur16, Rem. 4.8.5.17] for a treatment in the presentable setting). We conclude that

$$\mathcal{U}_{\text{wadd}}(\text{Perf}(A)) \rightarrow \left( \mathcal{U}_{\text{wadd}}(\text{Perf}(B)) \rightrightarrows \mathcal{U}_{\text{wadd}}(\text{Perf}(B \otimes_A B)) \rightrightarrows \dots \right)$$

is an  $\varepsilon$ -nilpotent limit diagram in  $\text{Mot}_{\kappa}(A)$ . The given  $F$  factors through an exact functor  $\text{Mot}_{\kappa}(A) \rightarrow \mathcal{D}$ , and we conclude by Proposition 2.18, part (4) and Proposition 2.19, part (2).  $\square$

**Remark 5.2.** We note that if the hypotheses of Theorem 5.1 are satisfied for  $A \rightarrow B$ , then they are satisfied for  $A' \rightarrow B \otimes_A A'$  for any  $\mathbb{E}_{\infty}$ -ring  $A'$  under  $A$ .

**5.1. Algebraic examples.** We now show that the desired hypothesis of rational surjectivity in Theorem 5.1 is satisfied for a finite flat extension of  $\mathbb{E}_{\infty}$ -ring spectra. In particular, algebraic  $K$ -theory, after any periodic localization, is a sheaf on the finite flat site.

We need a lemma first.

**Lemma 5.3.** Let  $R$  be a commutative discrete ring and let  $P$  be a perfect complex of  $R$ -modules whose Euler characteristic at every point of  $\text{Spec}(R)$  is nonzero. Then the class of  $P$  in  $K_0(R) \otimes \mathbb{Q}$  is a unit. If the Euler characteristic of  $P$  is a constant  $n$ , then the class  $[P] - n$  is nilpotent.

More generally, the same holds with  $\text{Spec}(R)$  replaced by any quasi-compact quasi-separated scheme  $X$ .

*Proof.* By restricting to the connected components of  $X$ , we see that it suffices to assume that  $\chi(P)$  is constant. So assume  $\chi(P) = n$  at every point: we prove that  $[P] - n$  is nilpotent. Then Zariski-locally on  $X$  we have that the class of  $[P]$  equals  $n$  in  $K_0$ . Now recall that if  $B \times_A B'$  is a fiber product of ring spectra, then any class  $x \in \pi_0(B \times_A B')$  which maps to zero in both  $\pi_0 B$  and  $\pi_0 B'$  necessarily satisfies  $x^2 = 0$ . Applying this inductively to a Zariski cover of  $X$ , using that  $K$ -theory satisfies Zariski descent, we can conclude the argument.  $\square$

**Proposition 5.4.** Let  $A \rightarrow B$  be a morphism of  $\mathbb{E}_{\infty}$ -ring spectra. Suppose that:

- (1) The  $\pi_0 A$ -module  $\pi_0 B$  is faithfully flat, finite, and projective.
- (2) The natural map  $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_*(B)$  is an isomorphism.

Then the hypotheses of Theorem 5.1 are satisfied.

*Proof.* It is easy to see that  $B$  is a perfect  $A$ -module. Applying Remark 5.2, it suffices to prove this proposition after replacing  $A$  and  $B$  with their connective covers. In this case,  $K_0(A) = K_0(\pi_0 A)$ . The result now follows from Lemma 5.3.  $\square$

**Example 5.5.** Let  $f: X \rightarrow Y$  be a morphism of noetherian schemes. Consider the symmetric monoidal functor  $f^*: \text{Perf}(Y) \rightarrow \text{Perf}(X)$ .

Suppose  $f$  is projective and of finite Tor-dimension, so that  $f_*$  defines a functor  $f_*: \text{Perf}(X) \rightarrow \text{Perf}(Y)$ . Suppose further that  $f$  is surjective.

In this case, we claim that  $f_*$  defines a rational surjection  $K_0(X) \otimes \mathbb{Q} \twoheadrightarrow K_0(Y) \otimes \mathbb{Q}$ . Without loss of generality, we may assume  $Y$  connected. Choose a relatively ample line bundle  $\mathcal{O}(1)$  on  $X$ . Then  $f_*(\mathcal{O}(n))$  is a perfect complex of nonzero Euler characteristic on  $Y$  for  $n \gg 0$ . (Base-change to the local ring of a codimension-zero point to reduce to the case  $Y = \text{Spec}(B)$  with  $B$  artinian; then since  $f_*(\mathcal{O}(n))$  is a perfect complex of  $B$ -modules homologically concentrated in degree 0, the

Auslander-Buchsbaum formula implies that  $f_*(\mathcal{O}(n))$  is finite free. On the other hand, it is nonzero because its pullback surjects onto  $\mathcal{O}(n)$ . Therefore the class of  $f_*(\mathcal{O}(n))$  is invertible in  $K_0(Y) \otimes \mathbb{Q}$  by Lemma 5.3.

Thus, for any weakly additive invariant  $F$  of  $\mathbb{Z}$ -linear  $\infty$ -categories, we have that

$$F(\mathrm{Perf}(Y)) \rightarrow \left( F(\mathrm{Perf}(X)) \rightrightarrows F(\mathrm{Perf}(X \times_Y X)) \xrightarrow{\rightarrow} \right)$$

is an  $\varepsilon$ -nilpotent limit. We use [BZFN10, Thm. 1.2] (whose hypotheses are satisfied by [BZFN10, Prop. 3.19, Cor. 3.23]) to identify  $\mathrm{Perf}(X \times_Y X) \simeq \mathrm{Perf}(X) \otimes_{\mathrm{Perf}(Y)} \mathrm{Perf}(X)$  and so on for the higher terms.

Note that if  $f$  is not flat, then the fiber products such as  $X \times_Y X$  need to be interpreted in the derived sense. However, if  $\ell$  is a prime invertible on  $Y$ , then the  $\ell$ -completed  $K$ -theory will be the same whether we take the derived or ordinary fiber products. In fact, one uses Zariski descent to reduce to the affine case; there, by the group-completion theorem, we need to see that if  $A$  is a connective  $\mathbb{E}_\infty$ -ring with  $1/\ell \in A$  then  $BGL_d(A) \rightarrow BGL_d(\pi_0 A) \simeq \tau_{\leq 1} BGL_d(A)$  is a mod  $\ell$  homology equivalence. This follows from the fact that for  $n > 1$ ,  $\pi_n BGL_d(A) \cong (\pi_{n-1} A)^{d^2}$  is uniquely  $\ell$ -divisible.

**5.2. Rognes's Galois extensions.** We now obtain our general descent result for Galois extensions of structured ring spectra in the sense of Rognes [Rog08].

**Theorem 5.6.** Let  $A \rightarrow B$  be a  $G$ -Galois extension of  $\mathbb{E}_\infty$ -ring spectra where  $G$  is finite. Suppose that the image of the transfer map  $K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$  contains the unit. Then, for all  $n \geq 0$  and implicit primes, we have:

- (1) The natural maps of spectra

$$L_n^f K(A) \rightarrow L_n^f(K(B)^{hG}) \rightarrow (L_n^f K(B))^{hG},$$

are equivalences.

- (2) If  $\mathcal{M}$  is any  $A$ -linear  $\infty$ -category and  $F: \mathrm{Mod}_{\mathrm{Perf}(A)}^\kappa(\mathrm{Cat}_\infty^{\mathrm{perf}}) \rightarrow \mathrm{Sp}$  is any weakly additive invariant for  $\kappa$  large enough, the maps

$$L_n^f F(\mathcal{M}) \rightarrow L_n^f(F(\mathcal{M} \otimes_A B)^{hG}) \rightarrow (L_n^f F(\mathcal{M} \otimes_A B))^{hG}$$

are equivalences.

- (3) Notation as above, the map

$$F(\mathcal{M}) \rightarrow (F(\mathcal{M} \otimes_A B))^{hG}$$

is an  $\varepsilon$ -equivalence (e.g.,  $K(A) \rightarrow K(B)^{hG}$  is an  $\varepsilon$ -equivalence).

- (4) The Tot tower that computes  $L_n^f F(\mathcal{M} \otimes_A B)^{hG}$  is quickly converging.

*Proof.* This follows from Theorem 5.1. We note that  $B$  is a dualizable, and hence perfect,  $A$ -module by [Rog08, Prop. 6.2.1], and that the equivalence  $B \otimes_A B \simeq \prod_{g \in G} B$  identifies the cobar construction  $B \rightrightarrows B \otimes_A B \xrightarrow{\rightarrow} \dots$  with the diagram computing the  $G$ -homotopy fixed points of  $B$ . Note also that each of the functors

$$B \mapsto \mathrm{Perf}(B) \mapsto \mathcal{U}_{\mathrm{wadd}}(\mathrm{Perf}(B)) \mapsto F(\mathrm{Perf}(B))$$

preserves finite products: in fact, any weakly additive invariant preserves finite products.  $\square$

We next observe that the condition of rational surjectivity of the transfer map is particularly transparent in the case of a Galois extension.

**Proposition 5.7.** Let  $G$  be a finite group and  $A \rightarrow B$  be a  $G$ -Galois extension. Then the following are equivalent:

- $K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$  has image containing the unit.
- The class of  $[B] \in K_0(A) \otimes \mathbb{Q}$  is a unit.
- The class  $[B] \in K_0(A) \otimes \mathbb{Q}$  is equal to  $|G|$ .

*Proof.* Clearly, the third condition implies the second, which implies the first. It suffices to argue that the first condition implies the third to prove that they are all equivalent.

Let  $i^*: K_0(A) \otimes \mathbb{Q} \rightarrow K_0(B) \otimes \mathbb{Q}$  be the canonical ring map and  $i_*: K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$  be the restriction map. Recall that if we regard  $K_0(B) \otimes \mathbb{Q}$  as a  $K_0(A) \otimes \mathbb{Q}$ -module via  $i^*$ , then  $i_*$  is a module map. Suppose  $x \in K_0(B) \otimes \mathbb{Q}$  has  $i_*(x) = 1$ . Then  $i^*$  is injective, since it has a section given by  $y \mapsto i_*(xy)$ . To show that  $[B] = |G|$ , it thus suffices to apply  $i^*$  and check it in  $K_0(B) \otimes \mathbb{Q}$ . We have

$$i^*[B] = [B \otimes_A B] = \left[ \prod_{g \in G} B \right] = |G|,$$

and we are done.  $\square$

For ease of reference, we give this property a name.

**Condition A.** A  $G$ -Galois extension  $A \rightarrow B$  satisfies the equivalent statements of Proposition 5.7.

We now give some examples.

**Example 5.8.** Let  $n \geq 1$  and let  $k$  be a field of characteristic zero containing the  $n$ th roots of unity. We let  $A$  be the unique  $\mathbb{E}_\infty$ -algebra over  $k$  with homotopy groups given by  $\pi_*(A) \simeq k[x_2^{\pm 1}]$  where  $|x_2| = 2$ . Here  $A$  can be obtained by starting with the free  $\mathbb{E}_\infty$ -algebra under  $k$  on a degree two class and then inverting it. There is a  $C_n$ -action on  $A$  (obtained, e.g., using this presentation) such that a generator acts on  $x_2$  by multiplication by a fixed primitive  $n$ th root of unity, and  $A' = (A)^{hC_n}$  has homotopy groups given by  $k[(x_2^n)^{\pm 1}]$ . The map  $A' \rightarrow A$  is a  $C_n$ -Galois extension and Condition A is satisfied, so we find that

$$K(A') \rightarrow K(A)^{hC_n}$$

is an  $\varepsilon$ -equivalence.

**Example 5.9.** Consider the  $C_2$ -Galois extension  $KO \rightarrow KU$ . In this case, the class in  $K_0(KO)$  of  $KU \in \text{Perf}(KO)$  is equal to 2, in view of Wood's theorem  $KO \wedge \Sigma^{-2}\mathbb{C}P^2 \simeq KU$ . Therefore, Condition A is satisfied and we find that the fiber of  $K(KO) \rightarrow K(KU)^{hC_2}$  is an  $\varepsilon$ -spectrum.

We note that the comparison problem here was raised in [AR12, Rem. 2.13]. In [BL14], it was shown that the map  $K(KO) \otimes \mathbb{Q} \rightarrow (K(KU) \otimes \mathbb{Q})^{hC_2}$  is an equivalence using localization sequences.

As indicated in the introduction, we can actually do better in this case.

**Theorem 5.10.** (1) The fiber of the comparison map  $K(KO) \rightarrow K(KU)^{hC_2}$  admits the structure of an  $\mathbb{F}_2$ -module spectrum.  
 (2) Let  $E_{p-1}$  denote Lubin-Tate  $E$ -theory at the height  $p-1$ , so that  $C_p$  is a subgroup of the Morava stabilizer group. Then, the fiber of the comparison map  $K(E_{p-1}^{hC_p}) \rightarrow K(E_{p-1})^{hC_p}$  admits the structure of an  $\mathbb{F}_p$ -module spectrum.  
 (3) More generally, if  $R \rightarrow R'$  is a  $G$ -Galois extension and the wrong-way map  $K_0(R') \rightarrow K_0(R)$  has image containing a prime number  $p$ , then the fiber of the map  $K(R) \rightarrow K(R')^{hG}$  admits the structure of an  $\mathbb{F}_p$ -module.

*Proof.* We will prove the third claim and then explain why the first and second claims are special cases. For the third claim, we will apply Theorem 4.5.

To apply this result we will first show that if  $R \rightarrow R'$  is a  $G$ -Galois extension of  $\mathbb{E}_\infty$ -rings in the sense of Rognes (with  $G$  finite), then  $\mathrm{Perf}(R')$  is dualizable in  $\mathrm{Mod}_{\mathrm{Cat}_\infty^{\mathrm{perf}}}(\mathrm{Perf}(R))$ . By [AG14, Thm. 3.15] (originally due to [Toë12, Prop. 1.5] for simplicial commutative rings), this statement is equivalent to the assertion that  $R'$  is a *smooth and proper*  $R$ -algebra, that is:

- (1)  $R'$  is a perfect  $R$ -module, so all mapping  $R$ -module spectra in  $\mathrm{Perf}(R')$  are perfect as  $R$ -modules.
- (2)  $R'$  is perfect as an  $R' \otimes_R R'$ -module.

The first condition is a general fact about Galois extensions: the perfect  $R$ -modules are the dualizable  $R$ -modules and  $R'$  is dualizable by [Rog08, Prop. 6.2.1]. The second condition follows from the equivalence of algebras  $R' \otimes_R R' \simeq \prod_{g \in G} R'$ , which implies that  $R'$  is a direct factor of a free  $R' \otimes_R R'$ -module of rank one.

Suppose  $R \rightarrow R'$  is a  $G$ -Galois extension and the image of  $K_0(R') \rightarrow K_0(R)$  contains the prime  $p$ . In this case, we find that the fiber of the map in  $\mathrm{Mot}_{\mathbb{N}_0}(R)$

$$\mathcal{U}_{\mathrm{wadd}}(\mathrm{Perf}(R)) \rightarrow \mathrm{Tot} \left( \mathcal{U}_{\mathrm{wadd}}(\mathrm{Perf}(R')) \rightrightarrows \mathcal{U}_{\mathrm{wadd}}(\mathrm{Perf}(R' \otimes_R R')) \rightrightarrows \dots \right)$$

or equivalently

$$\mathrm{fib}(\mathcal{U}_{\mathrm{wadd}}(\mathrm{Perf}(R)) \rightarrow \mathcal{U}_{\mathrm{wadd}}(\mathrm{Perf}(R'))^{hG})$$

admits the structure of an  $\mathbb{F}_p$ -module in  $\mathrm{Mot}_{\mathbb{N}_0}(R)$ . We now apply the functor

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathbb{N}_0}(R)}(\mathbf{1}, \cdot): \mathrm{Mot}_{\mathbb{N}_0}(R) \rightarrow \mathrm{Sp}$$

(which is continuous and cocontinuous) and use the identification (cf. [BGT13, Thm. 7.13] when  $R = S^0$  and [HSS15, Thm. 5.25] for  $R$  arbitrary) with  $K$ -theory to conclude that

$$\mathrm{fib}(K(R) \rightarrow K(R')^{hG})$$

has the structure of an  $\mathbb{F}_p$ -module spectrum.

Finally, we need to check that the relevant prime  $p$  belongs to the image of the transfer map in the cases of  $KO \rightarrow KU$  and  $E_{p-1}^{hC_p} \rightarrow E_{p-1}$ . We already observed this for the extension  $KO \rightarrow KU$  in Example 5.9. The claim for the extensions  $E_{p-1}^{hC_p} \rightarrow E_{p-1}$  is Corollary B.6.  $\square$

**5.3. Further Galois examples.** In this subsection, we will give several further examples of our Galois descent results. We will need various tools for verifying Condition A. We begin with the observation that if we can control  $K_0$  of a ring spectrum, checking Condition A (on an arbitrary Galois extension) is often straightforward.

We consider the following condition on a ring spectrum. Given an  $\mathbb{E}_\infty$ -ring  $A$ , let  $\mathrm{Idem}(A)$  be the set of idempotents in  $\pi_0(A)$ . We obtain a map  $\mathrm{Idem}(A) \rightarrow \mathrm{Idem}(K(A) \otimes \mathbb{Q})$  that sends an idempotent  $e \in \pi_0 A$  to the class of the perfect  $A$ -module  $A[e^{-1}]$ .

**Condition B.** The map  $\mathrm{Idem}(A) \rightarrow \mathrm{Idem}(K(A) \otimes \mathbb{Q})$  is bijective.

We first show that the map is often injective.

**Lemma 5.11.** Let  $A$  be an  $\mathbb{E}_\infty$ -ring. Suppose that  $A$  has no nontrivial torsion idempotents (equivalently, for every nonzero idempotent  $e \in \pi_0(A)$ , the rationalization  $A[e^{-1}]_{\mathbb{Q}}$  is nontrivial). Then the map  $\mathrm{Idem}(A) \rightarrow \mathrm{Idem}(K(A) \otimes \mathbb{Q})$  is injective.



*Proof.* Note first that the map  $\text{Idem}(A) \rightarrow \text{Idem}(K(A) \otimes \mathbb{Q})$  is a map of Boolean algebras. It suffices to show the map has trivial kernel. Let  $e$  be a nontrivial idempotent. We need to show that the class of the perfect  $A$ -module  $A[e^{-1}]$  is nontrivial in  $K_0(A) \otimes \mathbb{Q}$ . However, under the map  $K_0(A) \otimes \mathbb{Q} \rightarrow K_0(A[e^{-1}]) \otimes \mathbb{Q}$ , this class is carried to the unit. Therefore, it suffices to show that  $K_0(A[e^{-1}]) \otimes \mathbb{Q} \neq 0$ . This follows from the existence of a multiplicative trace map

$$K_0(A[e^{-1}]) \otimes \mathbb{Q} \rightarrow THH_0(A[e^{-1}]_{\mathbb{Q}}) \rightarrow \pi_0 A[e^{-1}]_{\mathbb{Q}}.$$

Since the target is nonzero by assumption, the source must be nonzero too.  $\square$

Condition B will play a basic role in this subsection, and most of our results will state that specific  $\mathbb{E}_{\infty}$ -ring spectra satisfy this condition. In particular, it (together with a mild statement about rationalization) implies the previous Condition A for every Galois extension.

**Proposition 5.12.** Suppose  $A$  is an  $\mathbb{E}_{\infty}$ -ring such that:

- (1)  $A$  satisfies Condition B.
- (2)  $A$  has no nontrivial torsion idempotents.

Then every  $G$ -Galois extension  $A \rightarrow B$  satisfies Condition A.

*Proof.* We use the notation in the discussion immediately following Condition A. Namely, we let  $i_*: K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$  the rational restriction of scalars map and  $i^*: K_0(A) \otimes \mathbb{Q} \rightarrow K_0(B) \otimes \mathbb{Q}$  be the usual map. Let  $y = i_*(1)/|G| \in K_0(A) \otimes \mathbb{Q}$ . We need to argue that  $y = 1$ .

We have  $y^2 = y$  by the relation  $B \otimes_A B \simeq \prod_G B$ , so  $y \in K_0(A) \otimes \mathbb{Q}$  is an idempotent. By Condition B, every idempotent in  $K_0(A) \otimes \mathbb{Q}$  arises from an idempotent  $e \in \pi_0 A$ . Consider the  $\mathbb{E}_{\infty}$ -ring  $A' = A[(1 - e)^{-1}]$  and the  $G$ -Galois extension  $A' \rightarrow B' \simeq B \otimes_A A'$ . Since  $A'$  is a perfect  $A$ -module, base-change along  $A \rightarrow A'$  preserves the Galois condition. It follows that we have a  $G$ -Galois extension  $A' \rightarrow B'$  such that  $[B'] = 0 \in K_0(A') \otimes \mathbb{Q}$ . We will now show that  $A'_{\mathbb{Q}} = 0$ . The hypothesis (2) we have assumed implies that  $e = 1$ , and we will then be done.

The image of  $[B']$  under the map  $K_0(A') \otimes \mathbb{Q} \rightarrow K_0(B') \otimes \mathbb{Q}$  is given by  $|G|$  via the Galois property  $B' \otimes_{A'} B' \simeq \prod_G B'$ . Thus, if  $[B'] \in K_0(A') \otimes \mathbb{Q}$  vanishes, we conclude that  $K_0(B') \otimes \mathbb{Q}$  is the zero ring and, by the argument with the trace of Lemma 5.11, we find that  $B'_{\mathbb{Q}} \simeq 0$ . By Lemma 5.13, we get that  $A'_{\mathbb{Q}} \simeq 0$ .<sup>4</sup>  $\square$

**Lemma 5.13.** Let  $B$  be an  $\mathbb{E}_{\infty}$ -ring. Suppose the finite group  $G$  acts on  $B$  in the  $\infty$ -category of  $\mathbb{E}_{\infty}$ -rings and set  $A = B^{hG}$ . Suppose an integer  $N$  is nilpotent in  $\pi_0(B)$ . Then  $N$  is nilpotent in  $\pi_0(A)$  too.

*Proof.* First of all,  $B$  splits into a direct product of its  $p$ -completions for  $p \mid N$ , and the  $p$ -completion  $\hat{B}_p$  has the property that it is annihilated by a power of  $p$ . The splitting is compatible with the action of  $G$ . Therefore, we may assume that  $N = p$  is a prime number.

Let  $G_p$  be a  $p$ -Sylow subgroup of  $G$ . Then  $A$  is a retract of  $B^{hG_p}$ . It suffices to show that  $p$  is nilpotent in  $B^{hG_p}$ . Therefore, we can reduce us to the case where  $G = G_p$  is itself a  $p$ -group. Since  $p$ -groups are nilpotent, an induction reduces to the case of  $G = C_p$ , which we now consider.

So suppose  $B$  is an  $\mathbb{E}_{\infty}$ -ring with a  $C_p$ -action and  $A \simeq B^{hC_p}$ . Suppose  $p$  is nilpotent in  $\pi_0(B)$ . By [BH15, Thm. 1.4], the  $C_p$ -action on  $B$  determines a cofree or Borel-complete genuine commutative  $C_p$ -ring spectrum  $R$  with  $R^{\{1\}} \simeq B$ ,  $R^{C_p} \simeq A$ . By the results of [Bru07], its  $\pi_0$  is therefore endowed with the structure of a Tambara functor [Tam93]. Therefore, we can consider the multiplicative

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<sup>4</sup>When  $A' \rightarrow B'$  is a faithful Galois extension, then  $A' \rightarrow B'$  is descendable, so that  $B'_{\mathbb{Q}} \simeq 0$  implies  $A'_{\mathbb{Q}} \simeq 0$  by descent. The use of Lemma 5.13 is to cover the non-faithful case.

norm  $N_e^{C_p}: \pi_0(B) \rightarrow \pi_0(A)$  as well as the additive norm  $\text{Tr}_e^{C_p}: \pi_0(B) \rightarrow \pi_0(A)$ . We have, using [Tam93, p. 1398, (v)],

$$N_e^{C_p}(p) = p + (p^{p-1} - 1)\text{Tr}_e^{C_p}(1) \in \pi_0(A).$$

Note that  $x = \text{Tr}_e^{C_p}(1)$  satisfies  $x^2 = px$ . Since for  $n \geq 0$  sufficiently large,  $p^n x = \text{Tr}_e^{C_p}(p^n) = 0$ , it follows that  $x$  is nilpotent: in fact,  $x^{n+1} = 0$ . Since  $N_e^{C_p}(p)$  is nilpotent in  $\pi_0(A)$  too, it follows that  $p$  is nilpotent in  $\pi_0(A)$ .  $\square$

We next exhibit a class of  $\mathbb{E}_\infty$ -rings  $A$  for which  $K_0(A)$  admits a purely algebraic description, and as a result, the above conditions are more amenable for them. In particular, we will show that in this case Condition B is satisfied.

**Proposition 5.14.** Let  $A$  be an  $\mathbb{E}_\infty$ -ring spectrum. Suppose  $\pi_*(A)$  is even, regular, and noetherian of finite Krull dimension. Then we have an isomorphism of commutative rings

$$K_0(A) \simeq K_0(\pi_*(A)),$$

where we consider  $\pi_*(A)$  as an ungraded ring.

*Proof.* We define the morphism

$$K_0(A) \rightarrow K_0(\pi_*(A))$$

(which by regularity is the Grothendieck group of finitely generated  $\pi_*(A)$ -modules) as follows. Given a perfect  $A$ -module  $M$ , we consider  $\pi_*(M) \simeq \pi_{\text{even}}(M) \oplus \pi_{\text{odd}}(M)$  as a finitely generated  $\pi_*(A)$ -module and take the class  $[\pi_{\text{even}}(M)] - [\pi_{\text{odd}}(M)]$  in  $K_0$ . The long exact sequence in homotopy implies easily that this defines a map  $K_0(A) \rightarrow K_0(\pi_*(A))$  as desired.

To define the isomorphism in the opposite direction, let  $\mathcal{C}$  be the category of finitely generated, *evenly* graded projective  $\pi_*(A)$ -modules. To any object of  $\mathcal{C}$  we can associate a perfect  $A$ -module (given by a retract of a sum of shifts of  $A$  itself) and we easily obtain a multiplicative map

$$K_0(\mathcal{C}) \rightarrow K_0(A).$$

For any  $P \in \mathcal{C}$ , the class of  $P$  and its shift  $P(2)$  (defined such that  $P(2)_k = P_{k+2}$ ) map to the same class in  $K_0(A)$ . By a theorem of van den Bergh [VdB86] (cf. also [Haz16, Cor. 6.4.2], but applied to the graded ring  $B_*$  with  $B_* = A_{2*}$ ), we have an isomorphism of rings

$$(5.15) \quad K_0(\pi_*(A)) \simeq K_0(\mathcal{C}) / \langle [P] - [P(2)] \mid P \in \mathcal{C} \rangle,$$

so we obtain a map

$$K_0(\pi_*(A)) \rightarrow K_0(A).$$

It suffices now to show that the two maps  $K_0(A) \rightarrow K_0(\pi_*(A)) \rightarrow K_0(A)$  and  $K_0(\pi_*(A)) \rightarrow K_0(A) \rightarrow K_0(\pi_*(A))$  are the identity.

- (1) For the first claim, we first observe that  $K_0(A)$  is generated by classes corresponding to retracts of graded free  $A$ -modules. To see this, we fix a perfect  $A$ -module  $M$  and induct on the homological dimension of  $\pi_*(M)$  as a  $\pi_*(A)$ -module. If  $\dim_{\pi_*(A)} \pi_*(M) = 0$ , then  $\pi_*(M)$  is projective so that  $M$  itself is a retract of a graded free  $A$ -module. If  $\dim_{\pi_*(A)} \pi_*(M) > 0$ , then we choose a graded free  $A$ -module and a map  $F \rightarrow M$  inducing a surjection on homotopy. We have then a cofiber sequence  $M' \rightarrow F \rightarrow M$ , where  $\dim_{\pi_*(A)} \pi_*(M') = \dim_{\pi_*(A)} \pi_*(M) - 1$ . Using  $[M] = [F] - [M']$  and induction, we can conclude that  $[M]$  belongs to the subgroup of  $K_0(A)$  generated by the classes corresponding to retracts of graded free  $A$ -modules.

Suppose  $M$  is a retract of a graded free  $A$ -module. In this case, one checks directly the map  $K_0(A) \rightarrow K_0(\pi_*(A)) \rightarrow K_0(A)$  carries  $[M]$  to itself, as desired. Since the classes  $[M]$  generate  $K_0(A)$ , this completes the verification of this case.

- (2) For the second map, we use (5.15). Consider a class in  $K_0(\pi_*(A))$  represented by an evenly graded  $\pi_*(A)$ -module  $P_*$  which is projective. We can find an  $A$ -module  $P$  with  $\pi_*(P) \simeq P_*$ . The map  $K_0(\pi_*(A)) \rightarrow K_0(A)$  carries  $[P_*] \mapsto [P]$ . It follows easily that the composite  $K_0(\pi_*(A)) \rightarrow K_0(\pi_*(A))$  is the identity on  $[P_*]$ , and therefore in general.  $\square$

**Corollary 5.16.** Assume  $A$  is an  $\mathbb{E}_\infty$ -ring such that  $A_*$  is even, regular, noetherian and of finite Krull dimension. Then  $A$  satisfies Condition B.

*Proof.* Without loss of generality, we can assume that  $\pi_0(A)$  (and thus  $\pi_*(A)$ ) has no nontrivial idempotents, after splitting into finitely many factors. The augmentation map  $K_0(A) \simeq K_0(\pi_*(A)) \rightarrow \mathbb{Z}$  takes values in  $\mathbb{Z}$ , because  $\pi_0(A)$  is connected. By [Swa68, Prop. 10.2], its kernel is nilpotent, hence  $K_0(A) \otimes \mathbb{Q}$  is a nilpotent thickening of  $\mathbb{Q}$ , and does therefore not contain any non-trivial idempotents.  $\square$

Next, we include a basic tool that enables us to check Condition B via descent.

**Proposition 5.17.** Suppose  $A$  is an  $\mathbb{E}_\infty$ -ring. Suppose that there exists an  $\mathbb{E}_\infty$ - $A$ -algebra  $A'$  such that:

- (1)  $A'$  is perfect as an  $A$ -module and the map  $K_0(A') \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$  from restriction of scalars is surjective.
- (2)  $A$  is  $A'$ -complete.
- (3)  $A'$  satisfies Condition B.
- (4) Suppose that either  $A'$  has no nontrivial idempotents, or  $A'$  and  $A' \otimes_A A'$  have no nontrivial torsion idempotents.

Then  $A$  satisfies Condition B and has no nontrivial torsion idempotents.

*Proof.* We have, since  $A$  is  $A'$ -complete and  $A'$  is dualizable in  $\text{Mod}(A)$ , an equivalence

$$A \simeq \text{Tot} \left( A' \rightrightarrows A' \otimes_A A' \overset{\rightarrow}{\rightrightarrows} \dots \right).$$

We also know that we have an equivalence

$$K(A)_\mathbb{Q} \simeq \text{Tot} \left( K(A')_\mathbb{Q} \rightrightarrows K(A' \otimes_A A')_\mathbb{Q} \overset{\rightarrow}{\rightrightarrows} \dots \right).$$

as a (very) special case of Theorem 5.1: in fact, in this case the above augmented cosimplicial object is seen to be split. The functor  $\text{Idem}: \text{CAlg}(\text{Sp}) \rightarrow \text{Set}$  commutes with all limits (as it is corepresentable by the  $\mathbb{E}_\infty$ -ring  $S^0 \times S^0$ ), and we find that we have equalizer diagrams

$$\begin{aligned} \text{Idem}(A) &\rightarrow (\text{Idem}(A') \rightrightarrows \text{Idem}(A' \otimes_A A')) \\ \text{Idem}(K(A)_\mathbb{Q}) &\rightarrow (\text{Idem}(K(A')_\mathbb{Q}) \rightrightarrows \text{Idem}(K(A' \otimes_A A')_\mathbb{Q})). \end{aligned}$$

We have a natural transformation between them.

A diagram-chase now completes the proof. First of all, since  $\text{Idem}(A') \rightarrow \text{Idem}(K(A')_\mathbb{Q})$  is an isomorphism, we get that  $\text{Idem}(A) \rightarrow \text{Idem}(K(A)_\mathbb{Q})$  is at least *injective*. For example, suppose that  $A', A' \otimes_A A'$  have no nontrivial torsion idempotents. Since by assumption  $\text{Idem}(A') \rightarrow \text{Idem}(K(A')_\mathbb{Q})$  is an isomorphism, and since by Lemma 5.11 the map  $\text{Idem}(A' \otimes_A A') \rightarrow \text{Idem}(K(A' \otimes_A A')_\mathbb{Q})$  is injective, we can conclude that  $\text{Idem}(A) \rightarrow \text{Idem}(K(A)_\mathbb{Q})$  is an isomorphism. The case

where  $A'$  has no nontrivial idempotents is easier and we find that  $\text{Idem}(A) \simeq \text{Idem}(K(A)_{\mathbb{Q}}) \simeq \{0, 1\}$ .  $\square$

For our example with topological modular forms, we will need to check Condition B by Zariski localization on the base. We will use the following result.

**Corollary 5.18.** Suppose  $A$  is an  $\mathbb{E}_{\infty}$ -ring. Suppose every localization  $A_{(p)}$  at a prime  $p$  satisfies Condition B. Then  $A$  satisfies Condition B.

*Proof.* We consider the following two examples of sheaves with values in the  $\infty$ -category  $\text{CAlg}(\text{Sp}_{\geq 0})$  of connective  $\mathbb{E}_{\infty}$ -ring spectra on the Zariski site of  $\text{Spec}(\mathbb{Z})$ :

- (1)  $\mathcal{F}_1$  assigns to the open subset  $\text{Spec}(\mathbb{Z}[1/N])$  the  $\mathbb{E}_{\infty}$ -ring  $\tau_{\geq 0}A[1/N]$ .
- (2)  $\mathcal{F}_2$  assigns to the open subset  $\text{Spec}(\mathbb{Z}[1/N])$  the  $\mathbb{E}_{\infty}$ -ring  $K(A[1/N])_{\mathbb{Q}}$ . Zariski descent in  $K$ -theory (a special case of Nisnevich descent in  $K$ -theory, cf. Proposition A.13) implies that this is a sheaf of connective  $\mathbb{E}_{\infty}$ -ring spectra.

Recall that  $\text{Idem}: \text{CAlg}(\text{Sp}_{\geq 0}) \rightarrow \text{Set}$  commutes with limits, so we obtain sheaves of sets  $\text{Idem}(\mathcal{F}_1), \text{Idem}(\mathcal{F}_2)$ . We have a natural map of sheaves

$$\text{Idem}(\mathcal{F}_1) \rightarrow \text{Idem}(\mathcal{F}_2)$$

Our assumption (and the fact that  $\text{Idem}$  commutes with filtered colimits) implies that the map on stalks is an isomorphism, so we get that  $\text{Idem}(\mathcal{F}_1) \simeq \text{Idem}(\mathcal{F}_2)$ . Taking global sections, we find that  $A$  satisfies Condition B as desired.  $\square$

We will now need various tools that enable one to reduce checking Condition B via other types of localization and completion. The next lemma is well-known in the algebraic context (where  $x$  has degree zero) (cf. [TT90, Ex. 3.19.2]).

**Lemma 5.19.** Let  $A$  be an  $\mathbb{E}_{\infty}$ -ring and let  $x \in \pi_*(A)$  be a homogeneous elements. Then the diagram

$$\begin{array}{ccc} K(A) & \longrightarrow & K(A[x^{-1}]) \\ \downarrow & & \downarrow \\ K(\widehat{A}_x) & \longrightarrow & K(\widehat{A}_x[x^{-1}]). \end{array}$$

is homotopy cartesian in the  $\infty$ -category  $\text{Sp}_{\geq 0}$  of connective spectra.

*Proof.* Let  $\text{Perf}_{x\text{-tor}}(A)$  denote the  $\infty$ -category of perfect  $A$ -modules which are  $x$ -power torsion. The natural functor

$$(5.20) \quad \text{Perf}_{x\text{-tor}}(A) \rightarrow \text{Perf}_{x\text{-tor}}(\widehat{A}_x)$$

given by tensoring with  $\widehat{A}_x$  is an equivalence. In fact, any *perfect*  $x$ -power torsion  $A$ -module is automatically  $x$ -adically complete as  $x$  will act nilpotently on it.

Now comparing the fiber sequences of *connective* spectra (cf. [Bar16, Prop. 11.16])

$$\begin{array}{ccccc} K(\text{Perf}_{x\text{-tor}}(A)) & \longrightarrow & K(\text{Perf}(A)) & \longrightarrow & K(\text{Perf}(A[x^{-1}])) \\ \downarrow \simeq & & \downarrow & & \downarrow \\ K(\text{Perf}_{x\text{-tor}}(\widehat{A}_x)) & \longrightarrow & K(\text{Perf}(\widehat{A}_x)) & \longrightarrow & K(\text{Perf}(\widehat{A}_x[x^{-1}])), \end{array}$$

shows that we have a fiber square in  $\text{Sp}_{\geq 0}$  as desired.  $\square$

Using similar arguments as in the proof of Corollary 5.18, we find the following result:

**Proposition 5.21.** Let  $A$  be an  $\mathbb{E}_\infty$ -ring. Suppose  $x \in \pi_*(A)$  is a homogenous element. Suppose  $\widehat{A}_x, A[x^{-1}], \widehat{A}_x[x^{-1}]$  satisfy Condition B. Then  $A$  satisfies Condition B.

Using similar reasoning, one also has:

**Proposition 5.22.** Let  $A$  be an  $\mathbb{E}_\infty$ -ring. Suppose  $x_1, \dots, x_n \in \pi_*(A)$ . Suppose  $A/(x_1, \dots, x_n)$  is contractible. Suppose the  $\mathbb{E}_\infty$ -ring  $A[(x_{i_1} \dots x_{i_k})^{-1}]$  satisfies Condition B for any nonempty collection of indices  $i_1, \dots, i_k \in \{1, \dots, n\}$ . Then  $A$  satisfies Condition B.

*Proof.* We use induction on  $n$ . When  $n = 1$ , the assertion is obvious, so we assume  $n > 1$  and assume the result proved for  $n - 1$  replacing  $n$ . Let  $A'$  be the localization of  $A$  away from the perfect  $A$ -module  $\text{End}_{\text{Mod}(A)}(A/(x_1, \dots, x_{n-1}))$  (cf. [MNN15a, §3] for an exposition). We then have  $A'[x_i^{-1}] \simeq A[x_i^{-1}]$  for  $1 \leq i \leq n-1$  and  $A'/(x_1, \dots, x_{n-1}) = 0$ . In addition, the natural map  $A \rightarrow A'$  induces an equivalence  $A/x_n \simeq A'/x_n$ . As a result, the natural map  $\text{Perf}(A) \rightarrow \text{Perf}(A')$  restricts to an equivalence on  $x_n$ -power torsion objects. We therefore obtain a fiber square of connective spectra:

$$\begin{array}{ccc} K(A) & \longrightarrow & K(A') \\ \downarrow & & \downarrow \\ K(A[x_n^{-1}]) & \longrightarrow & K(A'[x_n^{-1}]) \end{array}$$

By induction, it follows that  $A'$  and  $A'[x_n^{-1}]$  satisfy Condition B: in fact, we consider the sequence  $x_1, \dots, x_{n-1} \in \pi_* A'$ . Therefore, the above fiber square implies that  $A$  satisfies Condition B.  $\square$

We now include our main examples involving  $TMF$ , the spectrum of topological modular forms (cf. [DFHH14] for a textbook reference). We first need a general lemma about even periodic derived stacks, (cf. [MM15, Section 2] for an exposition of this). Given an even periodic derived stack  $(X, \mathcal{O}^{\text{top}})$ , we let  $\omega$  denote the line bundle  $\pi_2 \mathcal{O}^{\text{top}}$  on  $X$ .

**Lemma 5.23.** Let  $\mathfrak{X} = (X, \mathcal{O}^{\text{top}})$  be a regular, noetherian even periodic derived Deligne-Mumford stack. Suppose that the map  $X \rightarrow B\mathbb{G}_m$  classifying  $\omega$  is quasi-affine. Then  $\mathcal{O}^{\text{top}}(X)$  satisfies Condition B. If  $X$  is flat over  $\text{Spec } \mathbb{Z}$ , then  $\mathcal{O}^{\text{top}}(X)$  has no nontrivial torsion idempotents.

*Proof.* The quasi-affineness hypothesis is equivalent to the assertion that there sections  $s_1, \dots, s_n \in H^0(X, \omega^{\otimes \bullet})$  such that  $X[s_i^{-1}]$  is affine over  $B\mathbb{G}_m$ , i.e., arises as the quotient of a  $\mathbb{G}_m$ -action (or grading) on the spectrum of a graded ring  $R_{i,*}$ , and such that the  $\{s_i\}$  have no common vanishing locus. Note that a power of each  $s_i$  survives the descent spectral sequence (cf. the argument of [MM15, Prop. 3.24]); by passage to such a power, we may assume that  $s_i$  arises from an element (which for convenience we still denote by  $s_i$ ) in  $\pi_{2n_i} \mathcal{O}^{\text{top}}(X)$ .

We observe that  $\mathcal{O}^{\text{top}}(X)[s_i^{-1}]$  arises as the sheaf of global sections of an even periodic derived stack of the form  $\text{Spec}(R_{i,*})/\mathbb{G}_m$  where  $R_{i,*}$  is a regular noetherian graded-commutative ring concentrated in even degrees. In particular, we get

$$\pi_*(\mathcal{O}^{\text{top}}(X))[s_i^{-1}] \simeq R_{i,*},$$

so by Corollary 5.16 we find that  $\mathcal{O}^{\text{top}}(X)[s_i^{-1}]$  satisfies Condition B. Similarly, for any collection of indices  $i_1, \dots, i_k$ , we have that  $\mathcal{O}^{\text{top}}(X)[(s_{i_1} \dots s_{i_k})^{-1}]$  satisfies Condition B. It follows that  $\mathcal{O}^{\text{top}}(X)$  satisfies Condition B in view of Proposition 5.22.

Finally, one sees that the idempotents of  $\mathcal{O}^{\text{top}}(X)$  correspond precisely to the idempotents in the discrete ring  $\Gamma(X, \mathcal{O}_X)$ , and by hypotheses the latter is torsion-free. In fact, on the étale site of  $X$ , we consider the following two presheaves of  $\mathbb{E}_\infty$ -rings: the first sends  $\text{Spec } R \rightarrow X$  to  $\mathcal{O}^{\text{top}}(\text{Spec } R)$  and the second to  $R$  itself. The idempotents of each are naturally identified. Since the functor  $\text{Idem}: \text{CAlg}(\text{Sp}) \rightarrow \text{Set}$  commutes with limits, we find the identification  $\text{Idem}(\Gamma(X, \mathcal{O}^{\text{top}})) \simeq \text{Idem}(\Gamma(X, \mathcal{O}_X))$ .  $\square$

Let  $M_{\text{ell}}$  denote the moduli stack of elliptic curves. Let  $\mathfrak{M}_{\text{ell}} = (M_{\text{ell}}, \mathcal{O}^{\text{top}})$  be the Goerss-Hopkins-Miller derived stack, i.e., the pair of  $M_{\text{ell}}$  together with the sheaf  $\mathcal{O}^{\text{top}}$  of even periodic, elliptic  $\mathbb{E}_\infty$ -ring spectra that they construct. For any étale morphism (assumed representable) of stacks  $X \rightarrow M_{\text{ell}}$ , we can form the  $\mathbb{E}_\infty$ -ring of sections  $\mathcal{O}^{\text{top}}(X)$ . For example, we have by definition  $\mathcal{O}^{\text{top}}(M_{\text{ell}}) = \text{TMF}$ .

**Theorem 5.24.** Given any étale separated morphism  $X \rightarrow M_{\text{ell}}$ , the  $\mathbb{E}_\infty$ -ring  $\mathcal{O}^{\text{top}}(X)$  satisfies Condition B and has no nontrivial torsion idempotents. In particular, any Galois extension of  $\mathcal{O}^{\text{top}}(X)$  satisfies Condition A.

*Proof.* The final claim follows from the first one together with Proposition 5.12. By Corollary 5.18, it suffices to prove Condition B for  $\mathcal{O}^{\text{top}}(X)_{(p)}$  for a prime number  $p$ . We consider three different cases.

- (1)  $p \geq 5$ . In this case, we have an equivalence of stacks  $(M_{\text{ell}})_{(p)} \simeq \mathbb{P}(4, 6)[\Delta^{-1}]$ . Note that  $X_{(p)}$  is quasi-affine over  $(M_{\text{ell}})_{(p)}$  by Zariski's main theorem [Gro66, Th. 8.12.6]. By Lemma 5.23, we find that  $\mathcal{O}^{\text{top}}(X)_{(p)}$  satisfies Condition B.
- (2)  $p = 3$ . In this case, we have an equivalence  $\text{TMF}_{(3)} \wedge C \simeq \text{TMF}_1(2)_{(3)}$  for a three cell complex  $C$  with even cells (cf. [Mat15a, Thm. 4.13]). The moduli stack  $Y = M_{\text{ell},1}(2)_{(3)}$  has the property that  $Y \rightarrow B\mathbb{G}_m$  is affine (cf. the explicit presentation in [Sto12, §7]). Note also that  $Y \rightarrow (M_{\text{ell}})_{(3)}$  is a finite étale cover. It follows that  $X_{(3)} \times_{M_{\text{ell}}} Y$  is quasi-affine over  $B\mathbb{G}_m$  and the map of  $\mathbb{E}_\infty$ -rings

$$\mathcal{O}^{\text{top}}(X)_{(3)} \rightarrow \mathcal{O}^{\text{top}}(X_{(3)} \times_{M_{\text{ell}}} Y)$$

exhibits the target as a perfect module over the source; in fact by 0-affineness of the derived moduli stack (cf. [MM15, Thm. 7.2])

$$\mathcal{O}^{\text{top}}(X_{(3)} \times_{M_{\text{ell}}} Y) \simeq \mathcal{O}^{\text{top}}(X_{(3)}) \otimes_{\text{TMF}_{(3)}} \text{TMF}_1(2)_{(3)} \simeq \mathcal{O}^{\text{top}}(X)_{(3)} \wedge C.$$

By Lemma 5.23,  $\mathcal{O}^{\text{top}}(X_{(3)} \times_{M_{\text{ell}}} Y)$  satisfies Condition B and has no nontrivial torsion idempotents. In addition,

$$\mathcal{O}^{\text{top}}(X_{(3)} \times_{M_{\text{ell}}} Y) \otimes_{\mathcal{O}^{\text{top}}(X_{(3)})} \mathcal{O}^{\text{top}}(X_{(3)} \times_{M_{\text{ell}}} Y) \simeq \mathcal{O}^{\text{top}}(Y \times_{M_{\text{ell}}} Y)$$

also has no nontrivial torsion idempotents. The complex  $C$  enables us to conclude that  $\mathcal{O}^{\text{top}}(X)_{(3)}$  satisfies Condition B and has no nontrivial torsion idempotents, by Proposition 5.17.

- (3)  $p = 2$ . Here we argue similarly as in (2), using an eight cell complex  $DA(1)$  and the equivalence  $\text{TMF}_{(2)} \wedge DA(1) \simeq \text{TMF}_1(3)_{(2)}$  due to Hopkins-Mahowald (see [Mat15a, § 4]).  $\square$

**Remark 5.25.** It seems remarkable that to establish Condition A for  $\text{TMF}$ , we need the existence of certain specific finite complexes. In particular, we do not know the analogous result for the finite Galois extensions provided by the theory of topological automorphic forms from [BL10]. In fact, we do not know a single example of a finite Galois extension which violates Condition A.

**Example 5.26.** Consider the Galois extension  $TMF[1/n] = \mathcal{O}^{\text{top}}(M_{\text{ell}}[\frac{1}{n}]) \rightarrow TMF(n)$  with Galois group  $GL_2(\mathbb{Z}/n)$  (cf. [MM15, Thm. 7.6]). It follows that the map

$$K(TM[1/n]) \rightarrow K(TM(n))^{hGL_2(\mathbb{Z}/n\mathbb{Z})}$$

becomes an equivalence after any periodic localization.

We now describe the analog of our results for the Hill-Lawson extension of the sheaf  $\mathcal{O}^{\text{top}}$  on the étale site of  $M_{\text{ell}}$ . Let  $M_{\text{ell}}$  denote the compactified moduli stack of elliptic curves. In [HL16], Hill and Lawson describe the *log-étale site* of  $M_{\text{ell}}$  and endow it with a sheaf  $\mathcal{O}^{\text{top}}$  of  $\mathbb{E}_{\infty}$ -ring spectra which restricts to the previous sheaf on the étale site of  $M_{\text{ell}}$ .

**Theorem 5.27.** Let  $(X, M_X) \rightarrow M_{\text{ell}}$  be a log-étale map. Then  $\mathcal{O}^{\text{top}}(X, M_X)$  satisfies Condition B and has no nontrivial torsion idempotents. For example, every finite Galois extension of the  $\mathbb{E}_{\infty}$ -ring spectra  $Tmf, Tmf_0(n), Tmf_1(n)$  ( $n \geq 1$ ) satisfies Condition A.

*Proof.* We use Proposition 5.21.

After inverting the modular form  $\Delta^{24} \in \pi_{576} Tmf$ , the log-étale site of  $M_{\text{ell}}$  is identified with the étale site of  $M_{\text{ell}}$ . It follows that  $\mathcal{O}^{\text{top}}(X, M_X)[\Delta^{-24}]$  satisfies Condition B in view of Theorem 5.24.

The  $\mathbb{E}_{\infty}$ -ring  $\mathcal{O}^{\text{top}}(X)$ , and more generally the sheaf  $\mathcal{O}^{\text{top}}$ , is defined in [HL16] by gluing together  $\mathcal{O}^{\text{top}}(X \times_{M_{\text{ell}}} M_{\text{ell}})$  and the completion at the cusp. In particular, we can also evaluate  $\mathcal{O}^{\text{top}}$  on log-étale morphisms to the completion. Now, we need to consider the completion  $\mathcal{O}^{\text{top}}(\hat{X}) = \mathcal{O}^{\text{top}}(\widehat{X, M_X})_{\Delta^{24}}$ . This is obtained by completing the stack  $X$  at the modular form  $\Delta$ . Note first that  $(\widehat{M_{\text{ell}}})_{\Delta} \simeq (\text{Spf}\mathbb{Z}[[q]])/C_2$  via the Tate curve and its automorphism given by  $-1$ . We consider the  $\Delta$ -completed log stack  $\hat{Y} = X \times_{M_{\text{ell}}} \text{Spf}\mathbb{Z}[[q]]$ , so that  $\hat{Y}$  is log-étale over the log-scheme  $(\text{Spf}\mathbb{Z}[[q]], q)$ . We claim first that  $\mathcal{O}^{\text{top}}(\hat{Y})$  is even periodic and  $\pi_0 \mathcal{O}^{\text{top}}(\hat{Y})$  is regular. This follows from the discussion [HL16, Cor. 2.19] of the log-étale site of  $(\mathbb{Z}[[q]], q)$  and the construction in [HL16, Sec. 5.1]. Finally, the map  $\mathcal{O}^{\text{top}}(\hat{X}) \rightarrow \mathcal{O}^{\text{top}}(\hat{Y})$  exhibits an equivalence

$$\mathcal{O}^{\text{top}}(\hat{X}) \wedge \Sigma^{-2}\mathbb{C}\mathbb{P}^2 \simeq \mathcal{O}^{\text{top}}(\hat{Y}),$$

by Wood's theorem, which implies that the  $C_2$ -action on  $\mathcal{O}^{\text{top}}(\hat{Y}) \wedge \Sigma^{-2}\mathbb{C}\mathbb{P}^2 \simeq KU[[q]] \wedge \Sigma^{-2}\mathbb{C}\mathbb{P}^2$  is given by the coinduced representation. Therefore, we find by descent (Proposition 5.17) along  $\mathcal{O}^{\text{top}}(\hat{X}) \rightarrow \mathcal{O}^{\text{top}}(\hat{Y})$  and  $\mathcal{O}^{\text{top}}(\hat{X})[q^{-1}] \rightarrow \mathcal{O}^{\text{top}}(\hat{Y})[q^{-1}]$  that  $\mathcal{O}^{\text{top}}(\hat{X}), \mathcal{O}^{\text{top}}(\hat{X})[q^{-1}]$  satisfy Condition B. Putting everything together, we find by Proposition 5.21 that  $\mathcal{O}^{\text{top}}(X)$  satisfies Condition B.  $\square$

**Example 5.28.** Let  $n$  be square-free. Then we have a  $(\mathbb{Z}/n)^{\times}$ -Galois extension  $Tmf_0(n) \rightarrow Tmf_1(n)$  under  $Tmf[1/n]$  [MM15, Thm. 7.12]. This Galois extension satisfies Condition A, so that the map

$$K(Tmf_0(n)) \rightarrow K(Tmf_1(n))^{h(\mathbb{Z}/n)^{\times}}$$

is an  $\varepsilon$ -equivalence.

Finally, for completeness we give an example of a torsion  $\mathbb{E}_{\infty}$ -ring that does not satisfy Condition B. We do not know any non-torsion examples.

**Example 5.29.** Let  $G$  be a nontrivial finite  $p$ -group and let  $k$  be a field of characteristic  $p$ . Then there is an identification between  $\text{Perf}(k^{tG})$  and the *stable module  $\infty$ -category* of finite-dimensional  $k[G]$ -modules modulo projectives (cf. [Kel94] or [Mat15c]). Using this, we can calculate  $K_0(k^{tG})$ .

Every finite-dimensional  $k[G]$ -representation has a finite filtration with subquotients given by the trivial representation, and the representation  $k[G]$  is identified with zero in the stable module  $\infty$ -category. This forces  $K_0(k^{tG}) \simeq \mathbb{Z}/r$  where  $r \mid |G|$ . We also have a homomorphism  $K_0(k^{tG}) \rightarrow \mathbb{Z}/|G|$  that sends a representation to its dimension modulo  $|G|$ . Since two representations become isomorphic in the stable module  $\infty$ -category if and only if they are *stably isomorphic* as representations (i.e., become isomorphic after adding free summands), it follows that  $K_0(k^{tG}) \simeq \mathbb{Z}/|G|$ . In particular,  $K_0(k^{tG}) \otimes \mathbb{Q} = 0$ .

**5.4. Non-Galois examples of descent.** In this subsection, we record a few additional examples where one has descent but which are not Galois.

**Example 5.30.** We consider the connective version of Example 5.9 above. Consider the map of  $\mathbb{E}_\infty$ -rings  $ko \rightarrow ku$ . Since  $ku \simeq ko \wedge \Sigma^{-2}\mathbb{C}P^2$ , we observe that the class of  $ku$  in  $K_0(ko)$  is equal to 2. As a result, we conclude by Theorem 5.1 that we have an  $\varepsilon$ -nilpotent limit diagram given by the augmented cosimplicial object

$$K(ko) \rightarrow \left( K(ku) \rightrightarrows K(ku \wedge_{ko} ku) \xrightarrow{\rightarrow} \dots \right).$$

To compare this to Example 5.9, we consider the following diagram of localization sequences [BM08, BL14]:

$$\begin{array}{ccccc} K(\mathbb{Z}) & \xrightarrow{\quad} & K(ko) & \xrightarrow{\quad} & K(KO) \\ \downarrow & & \downarrow \simeq_\varepsilon & & \downarrow \simeq_\varepsilon \\ \mathrm{Tot}(F^\bullet) & \xrightarrow{\quad} & \mathrm{Tot}(K(ku^{\wedge_{ko} \bullet+1})) & \xrightarrow{\quad} & \mathrm{Tot}(K(KU^{\wedge_{KO} \bullet+1})) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ F(BC_{2+}, K(\mathbb{Z})) \simeq K(\mathbb{Z})^{hC_2} & \xrightarrow{\quad} & K(ku)^{hC_2} & \xrightarrow{\quad} & K(KU)^{hC_2} \end{array}$$

We have established the first two  $\varepsilon$ -equivalences in the first row of vertical arrows, which implies the induced map on the fibers is an  $\varepsilon$ -equivalence. It follows that  $K(ko) \rightarrow K(ku)^{hC_2}$  is an  $\varepsilon$ -equivalence if and only if the composite map  $K(\mathbb{Z}) \rightarrow F(BC_{2+}, K(\mathbb{Z}))$  is an  $\varepsilon$ -equivalence. However, by comparing with  $KU$  one sees that the map

$$K(\mathbb{Z}) \rightarrow K(\mathbb{Z})^{hC_2}$$

is not even a rational equivalence. To see this, we observe that the unit and the transfer of the unit from  $\pi_0 K(\mathbb{Z}) \otimes \mathbb{Q} \xrightarrow{\mathrm{Tr}} \pi_0(K(\mathbb{Z})^{hC_2}) \otimes \mathbb{Q}$  are linearly independent in  $\pi_0(K(\mathbb{Z})^{hC_2}) \otimes \mathbb{Q}$  because they are linearly independent in  $\pi_0(KU^{hC_2}) \otimes \mathbb{Q}$  and we have a  $K$ -theoretic map  $K(\mathbb{Z}) \rightarrow KU$ . As a result, we conclude that  $K(ko) \rightarrow K(ku)^{hC_2}$  is *not* an  $\varepsilon$ -equivalence (or even a rational equivalence). The same argument also shows that  $L_{K(1)}K(ko) \rightarrow (L_{K(1)}K(ku))^{hC_2}$  fails to be an equivalence when  $K(1)$  is the first Morava  $K$ -theory at the prime 2.

On the other hand, it is evident that if we rationalize before taking homotopy fixed points we obtain an equivalence  $K(\mathbb{Z}) \otimes \mathbb{Q} \rightarrow (K(\mathbb{Z}) \otimes \mathbb{Q})^{hC_2}$  and hence the vertical composites are all equivalences if we rationalize first and then take homotopy fixed points.

**Example 5.31.** Consider the map  $tmf[1/3] \rightarrow tmf_1(3)$ . We claim that this satisfies the assumptions of Theorem 5.1, so we have an  $\varepsilon$ -nilpotent limit diagram

$$K(tm f[1/3]) \rightarrow \left( K(tm f_1(3)) \rightrightarrows K(tm f_1(3) \wedge_{tm f[1/3]} tm f_1(3)) \xrightarrow{\rightarrow} \dots \right).$$



In fact, it suffices to show that the class  $[tmf_1(3)] - 8$  is nilpotent in  $K_0(tmf[1/3])$ . It suffices to check this after localizing at 2 and inverting 2. When localizing at 2, this follows from the complex  $DA(1)$  and the equivalence  $tmf_{(2)} \wedge DA(1) \simeq tmf_1(3)_{(2)}$  (cf. [Mat15a, §4]). After inverting 2, we see the equivalence just by considering homotopy.

## APPENDIX A. ÉTALE DESCENT FOR SPECTRAL ALGEBRAIC SPACES

We have seen in Proposition 5.4 that periodically localized algebraic  $K$ -theory (or indeed any periodically localized additive invariant) satisfies finite flat descent on  $\mathbb{E}_\infty$ -rings. In this appendix, we will describe the argument for Nisnevich descent for such invariants in the setting of spectral algebraic spaces as in [Lur, Ch. 3]. This argument is due to Thomason-Trobaugh [TT90], and we will indicate the necessary modifications in the present setting. Compare also the treatment by Barwick [Bar16, Prop. 12.12]. As a result, one obtains a basic étale descent result (Theorem A.14 below) which applies to invariants such as algebraic  $K$ -theory after periodic localization.

We need the definition of a *localizing invariant* in  $\text{Cat}_\infty^{\text{perf}}$ , as in [BGT13], although we shall not assume that our invariant commutes with filtered colimits. For simplicity, we will make the following definition in the rigid case.

**Definition A.1.** Let  $\mathcal{R} \in \text{CAlg}(\text{Cat}_\infty^{\text{perf}})$ . Suppose  $\mathcal{A}_1 \rightarrow \mathcal{A}_2$  is a morphism in  $\text{Mod}_{\mathcal{R}}(\text{Cat}_\infty^{\text{perf}})$  which is fully faithful on underlying  $\infty$ -categories. The *Verdier quotient*  $\mathcal{A}_2/\mathcal{A}_1$  is the pushout  $\mathcal{A}_2 \cup_{\mathcal{A}_1} \{0\}$  in  $\text{Mod}_{\mathcal{R}}(\text{Cat}_\infty^{\text{perf}})$ . A *weakly localizing invariant* of  $\mathcal{R}$ -linear  $\infty$ -categories with values in a presentable, stable  $\infty$ -category  $\mathcal{D}$  is a functor

$$F: \text{Mod}_{\mathcal{R}}(\text{Cat}_\infty^{\text{perf}}) \rightarrow \mathcal{D}$$

which carries Verdier quotient sequences to cofiber sequences in  $\mathcal{D}$ . It follows in particular that  $F$  is weakly additive in the sense of Definition 3.11.

Let  $X$  be a quasi-compact quasi-separated (hereafter *qcqs*) spectral algebraic space. Recall [Lur, Ch. 2] that one has a presentable, symmetric monoidal stable  $\infty$ -category  $\text{QCoh}(X)$  of *quasi-coherent sheaves* on  $X$ . By [Lur, Prop. 8.6.1.1], the  $\infty$ -category  $\text{QCoh}(X)$  is compactly generated, and the compact objects are given by the dualizable objects, which are denoted  $\text{Perf}(X)$ .

**Remark A.2.** The compact generation of  $\text{QCoh}(X)$  has a long history. For classical quasi-compact, separated schemes, the result is due to Neeman [Nee96, Prop. 2.5]. For classical qcqs schemes, the result appears in [BvdB03, Thm. 3.1.1]. That argument is extended to derived schemes in [BZFN10, Prop. 3.19].

Here  $\text{Perf}(X) \in \text{CAlg}(\text{Cat}_\infty^{\text{perf}})$ . Given a morphism  $f: Y \rightarrow X$  of qcqs spectral algebraic spaces, one obtains a symmetric monoidal pull-back functor  $f^*: \text{QCoh}(X) \rightarrow \text{QCoh}(Y)$  which restricts to dualizable or compact objects and yields a functor  $f^*: \text{Perf}(X) \rightarrow \text{Perf}(Y)$ .

The basic results will be fairly straightforward in the affine case. The general case will require a local-to-global argument for which the theory of *stable quasi-coherent stacks* of [Lur, Ch. 9] will be useful. A *stable quasi-coherent stack*  $\mathcal{C}$  on  $X$  assigns to every  $R$ -point  $\eta$  of  $X$  (for  $R$  a connective  $\mathbb{E}_\infty$ -ring) an  $R$ -linear presentable stable  $\infty$ -category  $\mathcal{C}_\eta$ . Given a map  $f: R \rightarrow R'$  of connective  $\mathbb{E}_\infty$ -rings, we obtain a new  $R'$ -point of  $X$  given by  $f^*\eta$  and we have a compatibility equivalence  $\mathcal{C}_{f^*\eta} \simeq \mathcal{C}_\eta \otimes_R R'$ . More precisely:

**Definition A.3** ([Lur, Ch. 9]). The  $\infty$ -category  $\mathrm{QStk}^{\mathrm{st}}(X)$  of stable quasi-coherent stacks is defined as  $\varprojlim_{\mathrm{Spec} R \rightarrow X} \mathrm{LinCat}_R^{\mathrm{st}}$  where  $\mathrm{LinCat}_R^{\mathrm{st}}$  denotes the  $\infty$ -category of presentable  $R$ -linear  $\infty$ -categories. Given a stable quasi-coherent stack  $\mathcal{C}$ , we can form the *global sections*  $\mathrm{QCoh}(X; \mathcal{C})$  which form a  $\mathrm{QCoh}(X)$ -linear presentable stable  $\infty$ -category.

Let  $X$  be a qcqs spectral algebraic space and let  $j: U \subset X$  be a quasi-compact open immersion. Let  $Z$  be the closed complement.

**Definition A.4.** Let  $\mathrm{QCoh}_Z(X) \subset \mathrm{QCoh}(X)$  be the subcategory of those  $\mathcal{F} \in \mathrm{QCoh}(X)$  such that  $j^*\mathcal{F} = 0$ . Let  $\mathrm{Perf}_Z(X) = \mathrm{QCoh}_Z(X) \cap \mathrm{Perf}(X)$  denote those perfect modules on  $X$  which restrict to zero on  $U$ .

We now show that  $\mathrm{QCoh}_Z(X)$  is compactly generated, with compact objects precisely  $\mathrm{Perf}_Z(X)$ . This will be relatively easy to check when  $X$  is affine. In general, the basic local-to-global observation that will be used throughout is that  $\mathrm{QCoh}_Z(X)$  arises from a stable quasi-coherent stack.

**Construction A.5.** We define the following three quasi-coherent stacks  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \mathrm{QStk}^{\mathrm{st}}(X)$  on  $X$ . Let  $\eta: \mathrm{Spec} R \rightarrow X$  be an  $R$ -point of  $X$  (with  $R$  a connective  $\mathbb{E}_\infty$ -ring).

- (1) We define  $\mathcal{C}_1$  via  $(\mathcal{C}_1)_\eta = \mathrm{Mod}(R) = \mathrm{QCoh}(\mathrm{Spec} R)$  itself. This is the unit in the  $\infty$ -category of stable quasi-coherent stacks. Clearly  $\mathrm{QCoh}(X; \mathcal{C}_1) \simeq \mathrm{QCoh}(X)$  by definition.
- (2) We define  $\mathcal{C}_2$  via  $(\mathcal{C}_2)_\eta = \mathrm{QCoh}(\mathrm{Spec} R \times_X U)$ . This is the push-forward of the unit in  $\mathrm{QStk}^{\mathrm{st}}(U)$  along  $j: U \rightarrow X$ . We have  $\mathrm{QCoh}(X; \mathcal{C}_2) \simeq \mathrm{QCoh}(U)$ .
- (3) Let  $(\mathcal{C}_3)_\eta \subset \mathrm{QCoh}(\mathrm{Spec} R) = \mathrm{Mod}(R)$  denote the subcategory of those quasi-coherent sheaves on  $\mathrm{Spec} R$  which restrict to zero on  $\mathrm{Spec} R \times_X U \subset \mathrm{Spec} R$ . Then  $\{(\mathcal{C}_3)_\eta\}_{\eta: \mathrm{Spec} R \rightarrow X}$  assembles into a quasi-coherent stack  $\mathcal{C}_3$ : in fact,  $\mathcal{C}_3$  is the pull-back  $\mathcal{C}_1 \times_{\mathcal{C}_2} 0$ . We note that limits in  $\mathrm{QStk}^{\mathrm{st}}(X)$  are computed pointwise (cf. [Lur, 9.1.3]). Unwinding the definitions again, we find that  $\mathrm{QCoh}(X; \mathcal{C}_3) \simeq \mathrm{QCoh}_Z(X)$ .

**Lemma A.6.** Suppose  $X = \mathrm{Spec}(A)$  is an affine spectral algebraic space and  $U \subset X$  a quasi-compact open subset of  $X$ , with closed complement  $Z$ . Then  $\mathrm{QCoh}_Z(X)$  is compactly generated.

*Proof.* This is proved in [Lur, Prop. 6.1.1.12, (e)]. The argument runs as follows. Choose elements  $f_1, \dots, f_k \in \pi_0 A$  such that  $U = \bigcup_i \mathrm{Spec}(A[f_i^{-1}])$ . Then  $\mathrm{QCoh}_Z(X)$  identifies with the full subcategory of  $\mathrm{QCoh}(X) = \mathrm{Mod}(A)$  spanned by those  $A$ -modules  $M$  such that  $M[f_i^{-1}] = 0$  for all  $i$ . One can take as compact generator the iterated cofiber  $A/(f_1, \dots, f_k)$ .  $\square$

The next result was known previously for classical qcqs schemes, see [Rou08, Thm. 6.8].

**Proposition A.7.** If  $X$  is a qcqs spectral algebraic space and  $j: U \rightarrow X$  a quasi-compact open immersion with  $Z$  the closed complement, then  $\mathrm{QCoh}_Z(X)$  is compactly generated. In addition, the inclusion  $\mathrm{QCoh}_Z(X) \rightarrow \mathrm{QCoh}(X)$  preserves compact objects.

*Proof.* We saw that  $\mathrm{QCoh}_Z(X)$  arises as the global sections of a stable quasi-coherent stack  $\mathcal{C}_3$ , which is locally compactly generated by Lemma A.6. By [Lur, Prop. 9.3.2.1], the global sections  $\mathrm{QCoh}_Z(X)$  are compactly generated. To see that  $\mathrm{QCoh}_Z(X) \rightarrow \mathrm{QCoh}(X)$  preserves compact objects, we note that both  $\infty$ -categories are compactly generated and the right adjoint to the inclusion is given by  $\mathcal{F} \mapsto \mathrm{fib}(\mathcal{F} \rightarrow j_* j^* \mathcal{F})$ , which commutes with filtered colimits (cf. [Lur, Prop. 2.5.4.3]). Therefore,  $\mathrm{QCoh}_Z(X) \rightarrow \mathrm{QCoh}(X)$  must preserve compact objects.  $\square$

**Corollary A.8.** Hypotheses as above, the sequence

$$(A.9) \quad \mathrm{Perf}_Z(X) \rightarrow \mathrm{Perf}(X) \xrightarrow{j^*} \mathrm{Perf}(U)$$

is a Verdier quotient sequence in  $\text{Mod}_{\text{Perf}(X)}(\text{Cat}_{\infty}^{\text{perf}})$ .

*Proof.* This follows from Proposition A.7 which implies that the Ind-completion of the above sequence is precisely

$$\text{QCoh}_Z(X) \rightarrow \text{QCoh}(X) \rightarrow \text{QCoh}(U).$$

Compare also [Lur, Prop. 6.2.3.1], which implies that we have a semi-orthogonal decomposition of  $\text{QCoh}(X)$ .  $\square$

**Corollary A.10** (Cf. [Bar16, Prop. 12.12]). Given any localizing invariant  $F: \text{Mod}_{\text{Perf}(X)}(\text{Cat}_{\infty}^{\text{perf}}) \rightarrow \mathcal{D}$ , we have a fiber sequence in  $\mathcal{D}$ ,

$$(A.11) \quad F(\text{Perf}_Z(X)) \rightarrow F(\text{Perf}(X)) \rightarrow F(\text{Perf}(U)).$$

We call (A.11) the *localization sequence*.

**Proposition A.12.** Let  $X, Y$  be qcqs spectral algebraic spaces and let  $j: U \subset X$  be a quasi-compact open immersion. Suppose  $f: Y \rightarrow X$  is a flat, representable morphism. Let  $Z$  be the reduced, discrete closed complement of  $j$  and suppose the map  $Y \times_X Z \xrightarrow{f \times_X \text{id}} Z$  is an equivalence. Then the map

$$f^*: \text{QCoh}_Z(X) \rightarrow \text{QCoh}_{f^{-1}(Z)}(Y)$$

is an equivalence of  $\text{Perf}(X)$ -linear  $\infty$ -categories.

*Proof.* We need to see that if  $M \in \text{QCoh}_Z(X)$ , then the map  $M \rightarrow f_* f^* M$  is an equivalence, and that if  $N \in \text{QCoh}_{f^{-1}(Z)}(Y)$  then  $f_* N \in \text{QCoh}_Z(X)$  and  $f^* f_* N \rightarrow N$  is an equivalence. Without loss of generality, we may assume that  $X = \text{Spec}(A)$  is affine and  $Y$  is a scheme. It suffices to check the desired claim on homotopy group sheaves. Since  $f$  is flat, the functor  $f^*$  simply tensors up over  $\pi_0$  on homotopy group sheaves. If  $M \in \text{QCoh}_Z(X)$ , then the homotopy groups of  $M$  are supported on  $f^{-1}(Z)$  and therefore have no higher derived pushforwards along  $f: Y \rightarrow X$ . As a result, the claim follows from the analog of our lemma in ordinary algebraic geometry, which is a special case of [TT90, Thm. 2.6.3].  $\square$

**Proposition A.13.** Let  $X$  be a qcqs spectral algebraic space,  $U$  a quasi-compact open subset of  $X$ ,  $f: Y \rightarrow X$  an étale map which is an isomorphism above  $Z = X \setminus U$ , and  $F$  a localizing invariant as above. Then the diagram

$$\begin{array}{ccc} F(\text{Perf}(X)) & \longrightarrow & F(\text{Perf}(U)) \\ \downarrow & & \downarrow \\ F(\text{Perf}(Y)) & \longrightarrow & F(\text{Perf}(Y \times_X U)) \end{array}$$

is a pullback square.

*Proof.* This will follow from the above localization sequence (A.11) provided we can show that pullback by  $f$  induces an equivalence  $\text{Perf}_Z(X) \simeq \text{Perf}_{f^{-1}(Z)}(Y)$ . This follows by taking compact objects in Proposition A.12.  $\square$

Finally, we can prove our main result.

**Theorem A.14.** Let  $X$  be a qcqs spectral algebraic space. Suppose that  $F: \text{Mod}_{\text{Perf}(X)}(\text{Cat}_{\infty}^{\text{perf}}) \rightarrow \mathcal{D}$  is a localizing invariant where  $\mathcal{D}$  is  $T(n)$ -local for some implicit prime  $p$  and height  $n$ . Then the presheaf

$$(U \rightarrow X) \mapsto F(\text{Perf}(U))$$

on the étale site  $X_{\text{ét}}$  of  $X$  is an étale sheaf.

*Proof.* Since Nisnevich excision is equivalent to Nisnevich descent by a theorem of Morel-Voevodsky (cf. [Lur, 3.7.5.1]), it follows from Proposition A.13 that  $U \mapsto F(\text{Perf}(U))$  is a Nisnevich sheaf when  $X$  is any qcqs spectral algebraic space. If  $X$  is an affine spectral scheme, the functor also satisfies descent for finite étale covers (Proposition 5.4). By [Lur, B.6.4.1], if  $X$  is affine, étale descent follows from Nisnevich descent and finite étale descent for affine schemes (Proposition 5.4).

To extend to the case of qcqs spectral algebraic spaces, we use [Lur, Ex. 3.7.1.5] to observe that  $X$  is Nisnevich-locally affine, and therefore the affine case implies the general one.  $\square$

**Remark A.15.** We note that the above result does not extend to the case where  $X$  is a Deligne-Mumford stack in general; counterexamples are easy to come by for the classifying stacks of finite groups. For example, one does not have descent for the  $C_2$ -Galois cover  $\text{Spec } \mathbb{C} \rightarrow (\text{Spec } \mathbb{C})/(C_2)$ , even rationally: we have

$$K_0((\text{Spec } \mathbb{C})/C_2) \simeq R(C_2) \simeq \mathbb{Z}[x]/(x^2 - x), \quad K_0(\text{Spec } \mathbb{C}) = \mathbb{Z}.$$

However, Theorem 5.24 together with the derived-affineness result of [MM15] implies that it nonetheless holds for the spectral Deligne-Mumford moduli stack  $\mathfrak{M}_{\text{ell}}$  underlying the theory of  $TMF$ , even though it fails for the underlying usual moduli stack  $M_{\text{ell}}$ .

## APPENDIX B. DESCENT FOR HIGHER REAL $K$ -THEORIES BY LENNART MEIER, JUSTIN NOEL, AND NIKO NAUMANN

In this appendix we record the existence of finite complexes with controlled Morava  $K$ -theory, as implicit in the work of Hopkins, Ravenel, and Smith. These are analogs of the results of Mitchell [Mit85] constructing finite complexes, the mod  $p$ -cohomology of which is finite free over certain finite sub-algebras of the Steenrod algebra. In our case, the algebra of operations replacing the Steenrod algebra is a group algebra of a Morava stabilizer group, and the finite sub-algebras are given by the group algebras of finite subgroups.

These results will in particular verify the assumption of our main results on descent for the Galois extensions afforded by higher real  $K$ -theories.

In order to formulate the results, we start with a reminder on Lubin-Tate theories, cf. [Rez97, GH05]. Fix a prime  $p > 0$ , a perfect field  $k$  of characteristic  $p$  and a (one-dimensional, commutative) formal group  $G$  of finite height  $n \geq 1$  over  $k$ .

Associated with this there is the automorphism group  $\text{Aut}(G, k)$  of the pair  $(G, k)$ , consisting of pairs  $(\alpha, \varphi)$  with an automorphism  $\alpha : k \rightarrow k$  and an isomorphism  $\varphi : \alpha^*G \rightarrow G$  of formal groups over  $k$ . There is an evident exact sequence of groups

$$(B.1) \quad 1 \rightarrow \text{Aut}_k(G) \rightarrow \text{Aut}(G, k) \rightarrow \text{Aut}(k),$$

the final map sending a pair  $(\alpha, \varphi)$  as above to  $\alpha$ . The central division algebra  $D$  over  $\mathbb{Q}_p$  of invariant  $1/n$  admits a unique maximal order  $\mathcal{O}_D \subset D$ , and its group of units  $\mathcal{O}_D^*$  is isomorphic to the automorphism group of the unique formal group of height  $n$  over  $\overline{\mathbb{F}}_p$ . Since  $G$  becomes isomorphic to this group over an algebraic closure of  $k$ , we can identify  $\text{Aut}_k(G) \subset \mathcal{O}_D^*$  as a closed subgroup.

The action of  $\text{Aut}_k(G)$  on the Lie-algebra of  $G$  over  $k$  affords a character, and we denote by  $\text{Aut}_k^1(G)$  its kernel:

$$1 \rightarrow \mathrm{Aut}_k^1(G) \rightarrow \mathrm{Aut}_k(G) \rightarrow k^*.$$

It is easy to see that  $\mathrm{Aut}_k^1(G)$  is a pro- $p$ -group, and hence the unique pro- $p$ -Sylow-subgroup of  $\mathrm{Aut}_k(G)$  because it is normal.

There is an even periodic  $\mathbb{E}_\infty$ -ring spectrum  $E(G, k)$  acted upon by  $\mathrm{Aut}(G, k)$  such that  $\pi_0 E(G, k) \cong W(k)[[u_1, \dots, u_{n-1}]]$  identifies with the universal deformation ring of  $G$  over  $k$ . This  $E(G, k)$  is Lubin-Tate theory. One can construct a map of  $\mathbb{E}_1$ -algebras  $E(G, k) \rightarrow K(G, k)$  which on  $\pi_0$  has the effect of quotienting out the maximal ideal [Ang08, Cor. 3.7]. The ring spectrum  $K(G, k)$  is even periodic with  $\pi_0 K(G, k) \cong k$ . This  $K(G, k)$  is the associated Morava  $K$ -theory. For every spectrum  $X$ ,  $K(G, k)^0(X)$  is canonically a continuous module over the completed twisted group-ring  $k[[\mathrm{Aut}(G, k)]]$  (twisted with respect to the action of  $\mathrm{Aut}(G, k)$  on  $k$  given by Equation (B.1)), and for every finite subgroup  $H \subset \mathrm{Aut}(G, k)$ , the twisted group ring  $k[H] \subset k[[\mathrm{Aut}(G, k)]]$  is a finite-dimensional sub-algebra.

Our existence result for finite complexes is the following.

**Theorem B.2.** For every finite subgroup  $H \subset \mathrm{Aut}(G, k)$ , there exists a finite,  $p$ -local complex  $X$  with cells in even dimensions such that  $K(G, k)^0(X)$  is a non-trivial, finite free  $k[H]$ -module.

Given a finite subgroup  $H \subset \mathrm{Aut}(G, k)$ , the spectrum  $E(G, k)$  is a (Borel complete) commutative algebra in genuine  $H$ -spectra, see for example [MNN15a, §6.3] for background on this. By a *semi-linear  $E(G, k)$ - $H$ -module*, we will mean an  $E(G, k)$ -module internal to Borel complete genuine  $H$ -spectra. The first example is  $M = E(G, k)$ , as is more generally  $M = E(G, k) \wedge X$  for any finite spectrum  $X$ , endowed with the  $H$ -action through the first smash factor.

The free example is the  $E(G, k)$ -module  $M = H_+ \wedge E(G, k)$ , endowed with the diagonal  $H$ -action. By the projection formula [MNN15a, Prop. 5.14], this is equivalent to  $\mathrm{Ind}_{\{e\}}^H \mathrm{Res}_{\{e\}}^H E(G, k)$ . Since  $H$  is finite, induction and coinduction agree, and given any semi-linear  $E(G, k)$ - $H$ -module  $N$ , the datum of a homotopy class of a semi-linear map  $\mathrm{map}(H, E(G, k)) \simeq \mathrm{Coind}_{\{e\}}^H \mathrm{Res}_{\{e\}}^H E(G, k) \rightarrow N$  is equivalent to the datum of the element of  $\pi_0 N$  obtained by evaluation at the unit.

We denote by  $DX = F(X, S)$  the Spanier-Whitehead dual of  $X$ .

**Corollary B.3.** In the situation of Theorem B.2, there is an equivalence of semi-linear  $E(G, k)$ - $H$ -modules  $E(G, k) \wedge DX \cong \mathrm{map}(H, E(G, k))^{\vee n}$  for some  $n \neq 0$ , and consequently there is an equivalence of  $E(G, k)^{hH}$ -modules  $E(G, k)^{hH} \wedge DX \cong E(G, k)^{\vee n}$ .

This result will very easily implies the next, which shows that the  $H$ -Galois extension  $A := E(G, k)^{hH} \rightarrow B := E(G, k)$  satisfies the assumption of Theorem 5.6.

**Corollary B.4.** In the situation of Corollary B.3, the rationalized transfer map

$$K_0(E(G, k)) \otimes \mathbb{Q} \rightarrow K_0(E(G, k)^{hH}) \otimes \mathbb{Q}$$

is surjective.

In some special cases, we can obtain stronger descent results in the algebraic  $K$ -theory of higher real  $K$ -theories, by using the following sharper variant of Theorem B.2. For this, we denote by  $X$  the  $p$ -local  $p$ -cell complex which has all attaching maps equal to  $\alpha_1$ , and which is called  $T(0)_{(1)}$  in [Rav86, Example 7.1.17].

**Theorem B.5.** Assume that  $n = p - 1$  and that  $\mathbb{F}_{p^n} \subset k$ . Then, for every  $C_p \subset \mathrm{Aut}_k(G)$ , the  $k[C_p]$ -module  $K(G, k)^0(X)$  is free of rank 1.

**Corollary B.6.** Assume that  $n = p - 1$  and that  $\mathbb{F}_{p^n} \subset k$ . Then  $p$  is in the image of the transfer map

$$K_0(E(G, k)) \rightarrow K_0(E(G, k)^{hC_p}).$$

The proof of Theorem B.5 is a direct application of Ravenel's computation of  $BP_*(X)$ . It will be explained at the very end of this appendix. We now begin working on the proof of Theorem B.2, while the proofs of Corollary B.3 and Corollary B.4 will appear immediately after that.

Our proof is a digest of some parts of [Rav92], and more exactly of J. Smith's construction of finite complexes [Smi]. Since the finite complex  $X$  will be constructed out of some complex projective space, we will first need some information about the  $k[[\text{Aut}(G, k)]]$ -module  $K(G, k)^0(\mathbb{CP}^\infty)$ . This is isomorphic to  $k[[T]]$ , after fixing a coordinate  $T$  of the formal group of the complex orientable ring spectrum  $K(G, k)$ . Writing  $F(X, Y) \in k[[X, Y]]$  for the resulting formal group law, we obtain

$$\text{Aut}_k(G) \cong \{f \in T \cdot k[[T]] \mid f'(0) \neq 0 \text{ and } F(f(X), f(Y)) = f(F(X, Y))\}.$$

Since  $K(G, k)^0(\mathbb{CP}^\infty) \cong k[[T]]$  is the ring of functions on our formal group, an element  $f \in \text{Aut}_k(G)$  acts on it as the unique continuous map of  $k$ -algebras sending  $T$  to  $f(T)$ . This completely determines the  $k[[\text{Aut}_k(G)]]$ -module  $K(G, k)^0(\mathbb{CP}^\infty)$ , and shows in particular that it is faithful. Jointly with  $K(G, k)^0(\mathbb{CP}^\infty) \cong \lim_N K(G, k)^0(\mathbb{CP}^N)$ , this implies the following.

**Proposition B.7.** For every  $1 \neq f \in \text{Aut}_k(G)$ , there is some  $N$  such that  $f$  acts non-trivially on  $K(G, k)^0(\mathbb{CP}^N)$ .

For the sake of readability, we now write  $\mathbb{CP}(N) := \mathbb{CP}^N$ .

**Proposition B.8.** There is some  $N \geq 0$  such that for every inclusion  $C_p \subset H \cap \text{Aut}_k(G)$ , we have

$$K(G, k)^0(\mathbb{CP}(N)^{\times(p-1)}) = U \oplus k[C_p]$$

as  $k[C_p]$ -modules, for some  $U$ .

*Proof.* Since  $H$  is finite, and using Proposition B.7, we can find some  $N \geq 0$  such that for every inclusion  $C_p \subset H$ , the  $k[C_p]$ -module  $V := K(G, k)^0(\mathbb{CP}(N))$  is non-trivial, and the claim then follows from  $K(G, k)^0(\mathbb{CP}(N)^{\times(p-1)}) \cong V^{\otimes(p-1)}$  and the following algebraic result, established during the proof of [Rav92, Theorem C.3.3]: If  $V$  is a non-trivial  $k[C_p]$ -module, then  $V^{\otimes(p-1)}$  splits off a free module of rank 1. (The proof in *loc. cit.* is written for some specific finite field, but evidently works for every field of characteristic  $p$ ).  $\square$

*Proof of Theorem B.2.* Put  $m := \dim_k(U) + 1$  with  $U$  as in Proposition B.8 and  $\kappa := (p-1)\binom{m+1}{2}$ . Then there is an idempotent  $e \in \mathbb{Z}_{(p)}[\Sigma_\kappa]$  such that for every finite-dimensional  $k$ -vectorspace  $W$ , the direct summand  $eW^{\otimes \kappa} \subset W^{\otimes \kappa}$  is non-zero if and only if  $\dim_\kappa(W) \geq m$  [Rav92, Theorem C.1.5]. In particular, we have  $eU^{\otimes \kappa} = 0$ .

With  $N \geq 0$  as in Proposition B.8, we now consider the complex

$$Y := e \cdot \left( \left( \mathbb{CP}(N)^{\times(p-1)} \right)^{\times \kappa} \right)_{(p)}.$$

It clearly is a finite,  $p$ -local complex with cells in even dimension.

We denote  $H'' := H \cap \text{Aut}_k^1(G)$  and claim that the  $k[H'']$ -module  $K(G, k)^0(Y)$  is (non-trivial and) finite free. Since  $H''$  is a  $p$ -group,  $k[H'']$  is an Artin local ring. By [Lam91, Thm. 19.29]  $K(G, k)^0(Y)$  is  $k[H'']$ -free if and only if it is  $k[H'']$ -projective. By Chouinard's theorem [Cho76, Cor. 1.1], this holds if for every inclusion  $E \subset H''$  of an elementary  $p$ -abelian subgroup, this module

is projective over  $k[E]$ . Every such  $E$  is a finite subgroup of the group of units of a commutative subfield of  $D$ , and thus cyclic. So we can assume that we are given some inclusion  $E = C_p \subset H''$ .

By construction, Proposition B.8 and the Künneth isomorphism, we have

$$K(G, k)^0(Y) = e \cdot ((U \oplus k[C_p])^{\otimes \kappa})$$

as a  $k[C_p]$ -module, using the notation above. Observing that every  $k[C_p]$ -module of the form  $k[C_p] \otimes M$  is free (say, as a consequence of the projection formula), we can multiply out

$$(U \oplus k[C_p])^{\otimes \kappa} = U^{\otimes \kappa} \oplus \tilde{F}$$

for some finite free  $k[C_p]$ -module  $\tilde{F} \neq 0$ . We can now conclude that

$$K(G, k)^0(Y) = e \cdot (U^{\otimes \kappa} \oplus \tilde{F}) = e \cdot \tilde{F}$$

is a non-trivial finite free  $k[C_p]$ -module, and hence finite free over  $k[H'']$ , as claimed.

We now want to induce this up along the inclusions of groups  $H'' \subset H' := H \cap \text{Aut}_k(G) \subset H$ . The  $k[H']$ -module  $K(G, k)^0(Y) \otimes_{k[H'']} k[H']$  is clearly finite free, and by the projection formula, it is isomorphic to  $K(G, k)^0(Y) \otimes_k k[H'/H'']$ , endowed with the diagonal  $H'$ -action. So we next find a finite even complex  $Z$  with  $K^0(G, k)(Z) \cong k[H'/H'']$  as a  $k[H']$ -module, for then  $K(G, k)^0(Y \wedge Z)$  will be finite free over  $k[H']$ .

Now, since  $\text{char}(k) = p$ , multiplication by  $p$  on  $k^*$  is injective, and hence  $H'/H'' \subseteq k^*$  is a cyclic group of order coprime to  $p$ . Since  $k[H'/H'']$  is semi-simple and  $k$  evidently contains the  $|H'/H''|$ -roots of unity, as an  $H'/H''$ -module  $k[H'/H'']$  is a sum of powers of a generating character. Pulling back to an  $H'$ -action and recalling our initial discussion, we have

$$k[H'/H''] \cong \bigoplus_{i=0}^{|H'/H''|-1} Lie^{\otimes i}.$$

We can thus take  $Z := \bigvee_{i=0}^{|H'/H''|-1} S^{2i}$  for the desired complex.

To induct up further along  $H' \subset H$ , we observe that

$$K(G, k)^0(Y \wedge Z) \otimes_{k[H']} k[H] \cong K(G, k)^0(Y \wedge Z) \otimes_k k[\Gamma],$$

where we denote  $\Gamma := H/H' \subset \text{Aut}(k)$ . As above, we then need to find a finite even complex  $W$  with  $K^0(G, k)(W) \cong k[\Gamma]$  as a  $k[H]$ -module, for then  $X := Y \wedge Z \wedge W$  will be as desired. By Galois descent, we have an isomorphism of semi-linear  $k - \Gamma$ -modules

$$k[\Gamma] \cong k \otimes_{k^\Gamma} k \cong k \otimes_{k^\Gamma} (k^\Gamma)^{\oplus |\Gamma|} \cong k^{\oplus |\Gamma|},$$

where the semi-linear action on the tensor products is through the left tensor factors. Hence we can take  $W := \bigvee^{\oplus |\Gamma|} S^0$  as the desired complex.  $\square$

*Proof of Corollary B.3.* Fix a basis  $\{\alpha_1, \dots, \alpha_n\}$  of the  $k[H]$ -module  $K^0(G, k)(X)$ . Since  $X$  is even, these lift to elements in  $E(G, k)^0(X) = \pi_0(E(G, k) \wedge DX)$  which determine a semi-linear map  $\text{map}(H, E(G, k))^{\vee n} \rightarrow E(G, k) \wedge DX$ , and it suffices to show that this map is an equivalence (of spectra). Since both spectra are finite free  $E(G, k)$ -modules, it suffices to check this after application of  $\pi_0(-) \otimes_{E^0(G, k)} K^0(G, k)$  which yields the  $K^0 - H$  semi-linear map  $\text{map}(H, K^0)^{\oplus n} \rightarrow K^0(X)$  determined by the basis above. This map is clearly an isomorphism.  $\square$

*Proof of Corollary B.4.* By Corollary B.3, there is a finite even complex  $X$  such that the  $E(G, k)^{hH}$ -module  $E(G, k)^{hH} \wedge DX$  admits the structure of an  $E(G, k)$ -module, hence the class  $[E(G, k)^{hH} \wedge DX] \in K_0(E(G, k)^{hH})$  lies in the image of the transfer map. Using  $[\Sigma^2 DX] = [DX]$ , that  $DX$  has

only even cells and induction on this number  $M$  of cells shows that  $[E(G, k)^{hH} \wedge DX] = M \cdot [E^{hH}]$  is a positive multiple of the unit. So the image of the rationalized transfer is an ideal which contains 1; hence the rationalized transfer is surjective.  $\square$

*Proof of Theorem B.5.* We need to compute the  $k[C_p]$ -module  $V := K(G, k)^0(X)$ , which is a  $p$ -dimensional representation of  $C_p$  over  $k$ . We will deduce this from Ravenel's result [Rav86, Lemma 7.1.11], which states that  $BP_*(X)$  is isomorphic, as a  $BP_*BP$ -comodule, to the subcomodule of  $BP_*BP \cong BP_*[t_1, t_2, \dots]$  generated freely over  $BP_*$  by the set  $\{t_1^i \mid 0 \leq i \leq p-1\}$ . This implies that the  $K(G, k)_0K(G, k)$ -comodule  $M := K(G, k)_0(X)$  is free over  $k$  on the images of the  $t_1^i$ .

Recall that we can identify

$$K(G, k)_0K(G, k) \cong \text{maps}_c(\text{Aut}_k^1(G), k) \cong k[t_1, t_2, \dots]/(t_i^{p^n} - t_i)$$

in such a way that the continuous map  $t_i$  on  $\text{Aut}_k^1(G)$  is given as follows. We have  $\text{Aut}_k^1(G) \subset 1 + \Pi \cdot \mathcal{O}_D$ , where  $D$  is the skew field over  $\mathbb{Q}_p$  of invariant  $1/n$ , and  $\Pi$  is a uniformizer. Since  $\mathbb{F}_{p^n} \subset k$ , this inclusion is in fact an equality, and we can write every  $g \in \text{Aut}_k^1(G)$  uniquely as  $g = 1 + \sum_i [t_i(g)]\Pi^i$  using the Teichmüller lift

$$[-] : \mathbb{F}_{p^n} \longrightarrow W(\mathbb{F}_{p^n}) \subset \mathcal{O}_D.$$

Briefly, the  $t_i$  are the digits in the  $\Pi$ -adic expansion.

To ease the notation, we introduce  $e_i := t_1^i$ , a  $k$ -basis of  $M$ , and denote by  $f_i \in V$  the dual basis. Writing  $\psi : M \longrightarrow K(G, k)_0K(G, k) \otimes_k M$  for the comodule structure, our representation  $V$  is the dual of  $M$ , in the sense that we have  $(g \cdot v)(m) = (\text{ev}_g \otimes v)(\psi(m))$  for every  $g \in \text{Aut}_k^1(G)$ ,  $v \in V$ ,  $m \in M$ , and denoting  $\text{ev}_g : K(G, k)_0K(G, k) \rightarrow k$  the evaluation of a continuous map at  $g$ . Finally, it is clear that the map  $t_1 : \text{Aut}_k^1(G) \longrightarrow (\mathbb{F}_{p^n}, +) \subset k$  is a homomorphism, and hence  $t_1$  is primitive, and in particular that implies that  $\psi(e_j) = (t_1 \otimes 1 + 1 \otimes e_1)^j$ . Given the above, a formal computation which we leave to the reader, gives

$$g \cdot f_i = f_i + \sum_{j>i}^{p-1} \binom{j}{i} \alpha^{j-i} \cdot f_j$$

for all  $g \in \text{Aut}_k^1(G)$ ,  $0 \leq i \leq p-1$  and we abbreviate  $\alpha := t_1(g)$ . For the endomorphism  $\varphi := g - 1 \in \text{End}_k(V)$ , this implies that  $\varphi^p = 0$  and that  $\varphi^{p-1}(f_0) = (p-1)!\alpha^{p-1}f_{p-1}$ . In particular, if  $\alpha \neq 0$ , then  $\varphi^{p-1} \neq 0$ .

We now specialize this to the case of interest, namely when  $\langle g \rangle = C_p$ , and claim that in this case we have  $\alpha = t_1(g) \neq 0$ . By the above reminder on the relation between the  $t_i$  and the  $\Pi$ -adic expansion of  $g$ , this is equivalent to saying that  $g - 1$  and  $\Pi$  have the same valuation in  $D$ . This is clear, because both valuations are  $\frac{1}{p-1}$  times the valuation of  $p$ :  $g$  is a primitive  $p$ th root of unity in  $D$  and  $1/n = 1/(p-1)$  is the invariant of  $D$ .

We conclude that the order of  $g - 1$  acting on  $V$  is exactly  $p$ , and the Jordan normal form implies, that  $V$  is a free module (of rank 1) over  $k[C_p]$ , as desired.  $\square$

*Proof of Corollary B.6.* This follows from Theorem B.5 in the same way in which Corollary B.4 follows from Theorem B.2.  $\square$



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